Chapter 9: Stress Transformation

Chapter Objectives

- ✓ Navigate between rectilinear coordinate systems for stress components
- ✓ Determine principal stresses and maximum in-plane shear stress
- ✓ Determine the absolute maximum shear stress in 2D and 3D cases

General stress state

The general state of stress at a point is characterized by

- three independent normal stress components $\sigma_x, \sigma_y,$ and σ_z
- three independent shear stress components $\tau_{xy}, \tau_{yz},$ and τ_{xz}

At a given point, we can draw a stress element that shows the normal and shear stresses acting on the faces of a small (infinitesimal) cube of material surrounding the point of interest

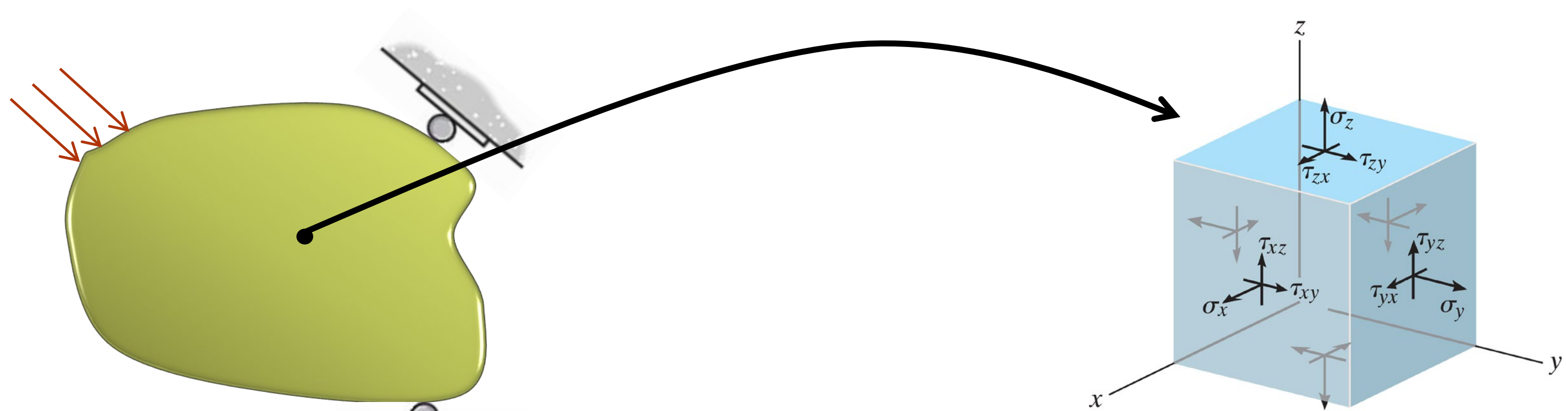


Figure: 01_11

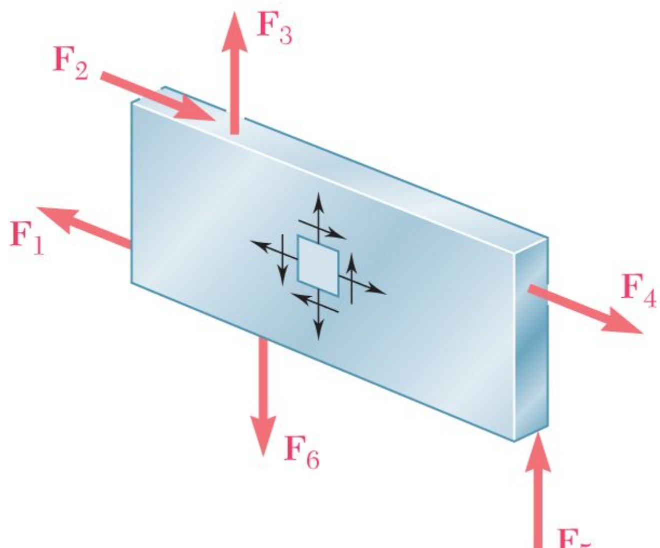
Plane Stress

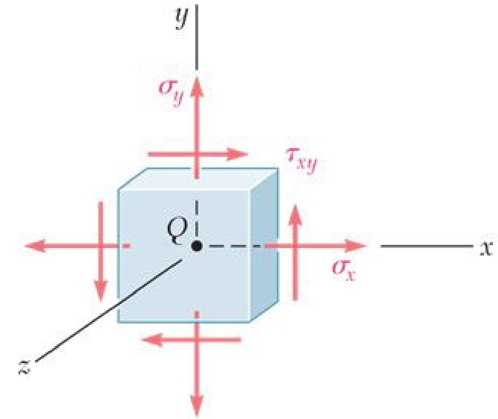
- Often, a loading situation involves only loads and constraints acting applied within a two-dimensional plane (e.g. the xy plane). In this case, any stresses acting in the third plane (z in this case) are equal to zero:

$$\sigma_z = \tau_{zx} = \tau_{zy} = 0$$

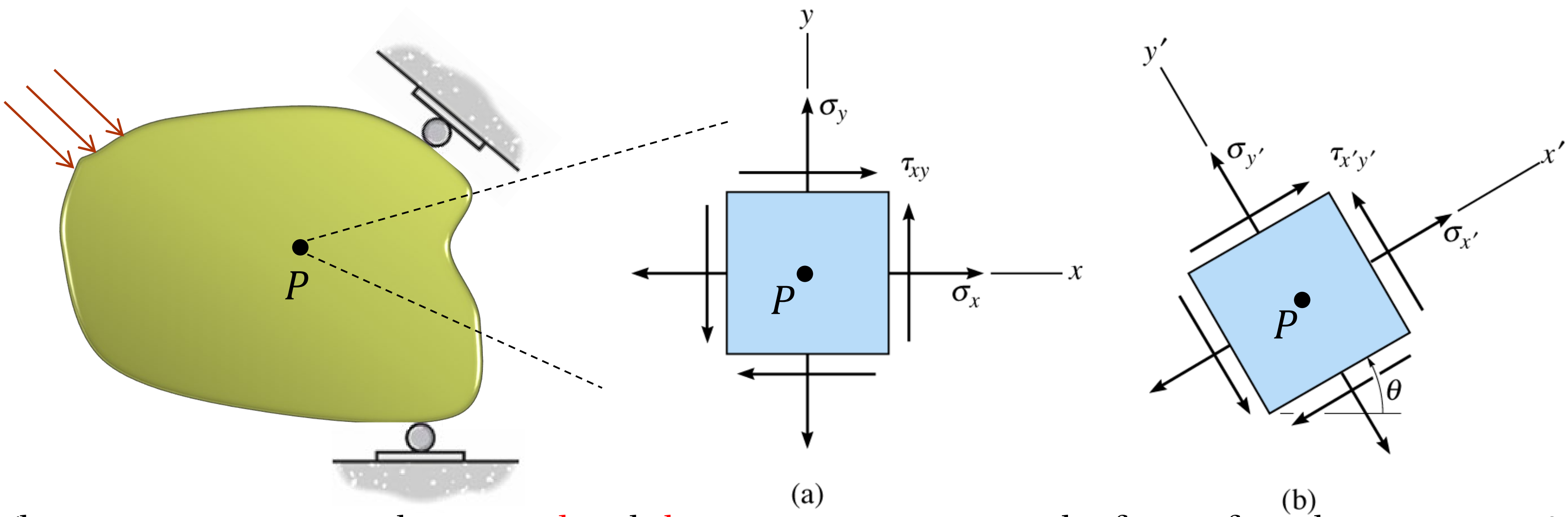
- Example:

Thin plates subject to forces acting in the mid-plane of the plate





Plane Stress Transformation

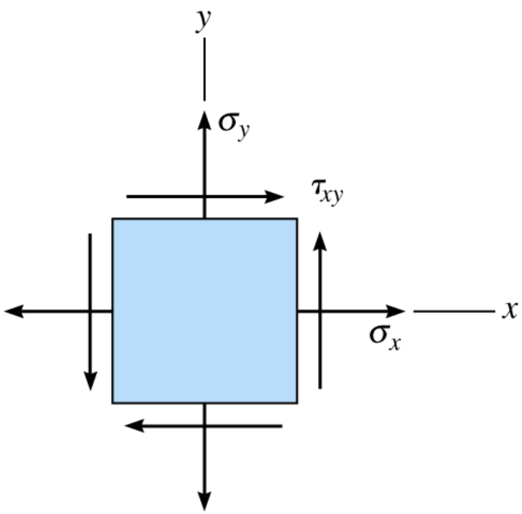


The stress tensor gives the normal and shear stresses acting on the faces of a cube (square in 2D) whose faces align with a particular coordinate system.

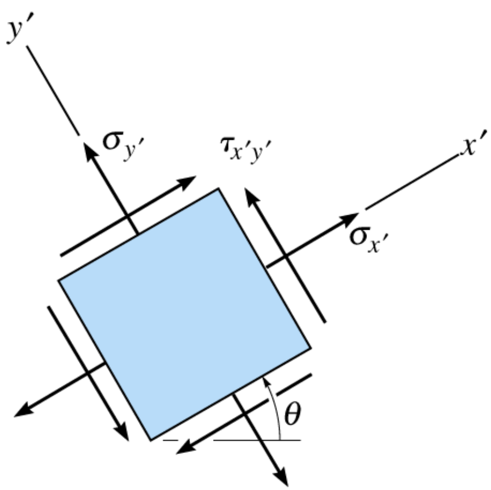
But, the <u>choice of coordinate system is arbitrary</u>. We are free to express the normal and shear stresses on any face we wish, not just faces aligned with a particular coordinate system.

Stress transformation equations give us a formula/methodology for taking known normal and shear stresses acting on faces in one coordinate system (e.g. x-y above) and converting them to normal and shear stresses on faces aligned with some other coordinate system (e.g. x’-y’ above)

Plane Stress Transformation



(a)



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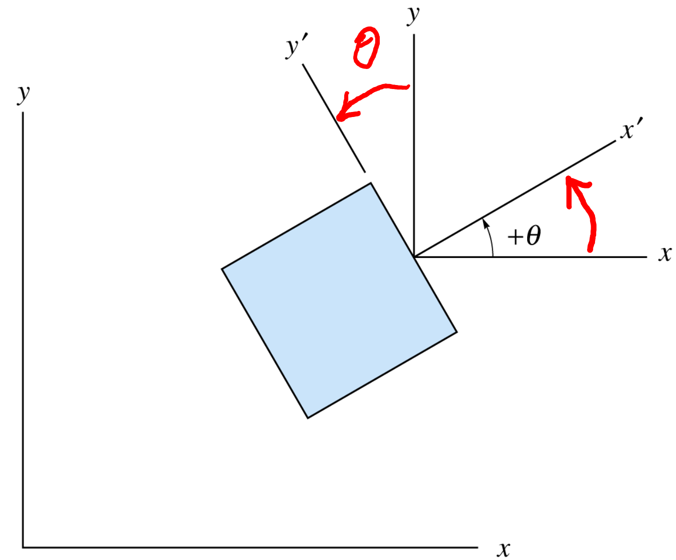
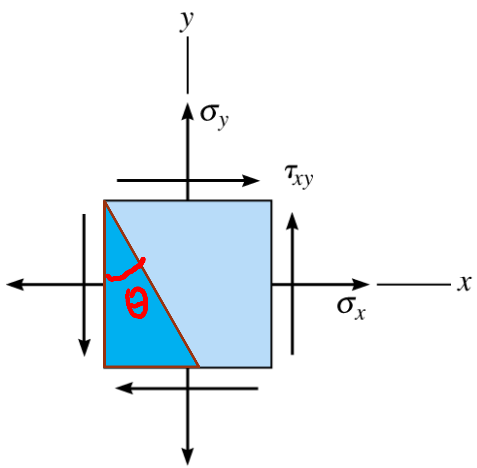
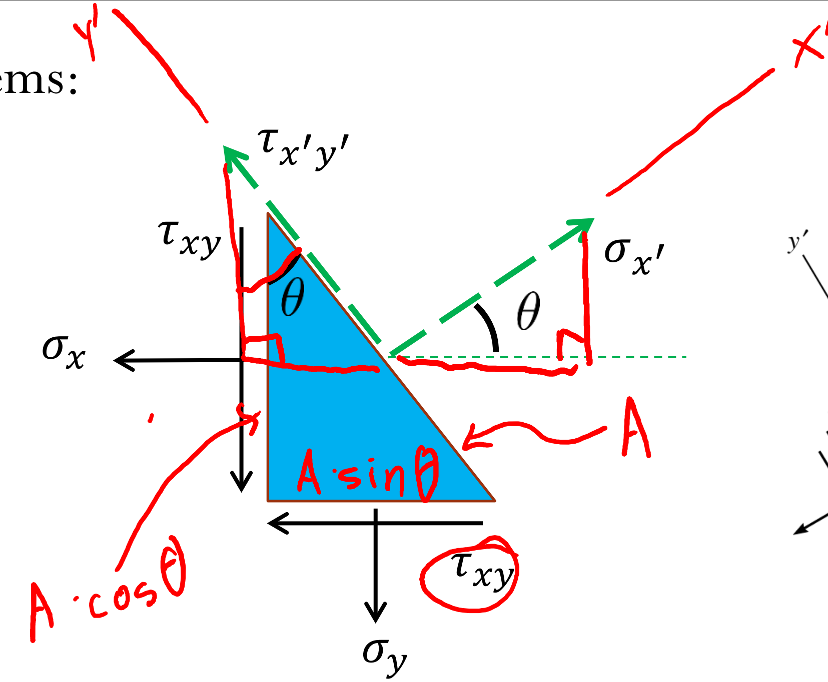


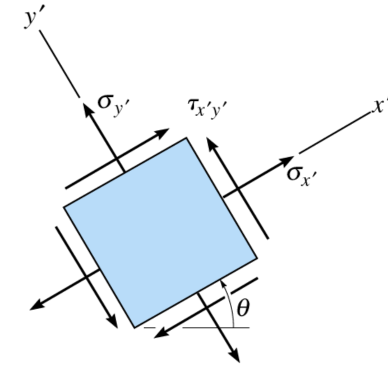
- Sign convention:
  - ➢ Both the x-y and x'-y' system follow the right-hand rule
  - ➢ The orientation of an inclined plane (on which the normal and shear stress components are to be determined) will be defined using the angle θ. The angle θ is measured from the positive x to the positive x'-axis. It is positive if it follows the curl of the right-hand fingers.

For two-dimensional problems:



(a)





(b)

$$\Sigma F_x = 0 \Rightarrow \sigma_x \cdot \cancel{A} \cdot \cos\theta - \tau_{xy} \cdot \cancel{A} \cdot \sin\theta + \sigma_{x'} \cdot \cos\theta \cdot \cancel{A} - \tau_{x'y'} \cdot \sin\theta \, \cancel{A} = 0$$

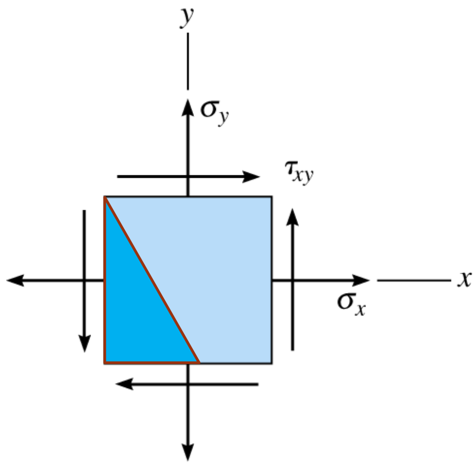
$$\sigma_x \cdot \cos\theta - \tau_{xy} \cdot \sin\theta + \underline{\sigma_{x'}} \cos\theta - \underline{\tau_{x'y'}} \sin\theta = 0$$

$$\Sigma F_y = 0 \Rightarrow -\sigma_y \cdot \cancel{A} \cdot \sin\theta - \tau_{xy} \cdot \cancel{A} \cdot \cos\theta + \underline{\sigma_{x'}} \cdot \sin\theta \cdot \cancel{A} + \underline{\tau_{x'y'}} \cdot \cos\theta \cdot \cancel{A} = 0$$

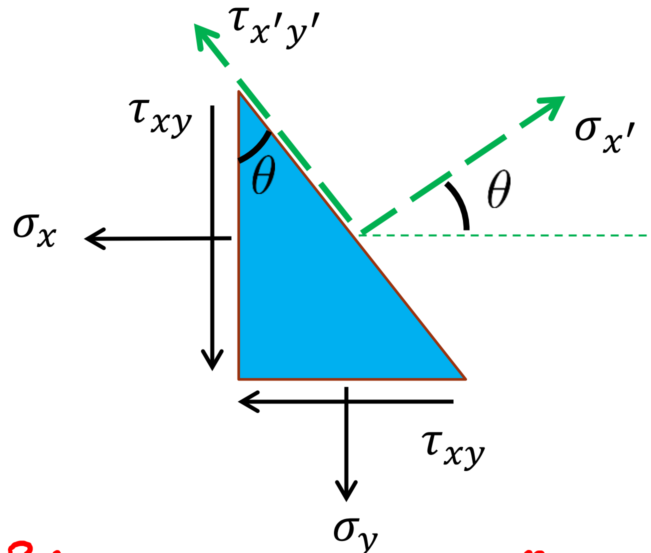
$$\sigma_{x'} \, \underline{\cos\theta} - \tau_{x'y'} \underline{\sin\theta} = -\sigma_x \cdot \cos\theta + \tau_{xy} \cdot \sin\theta$$

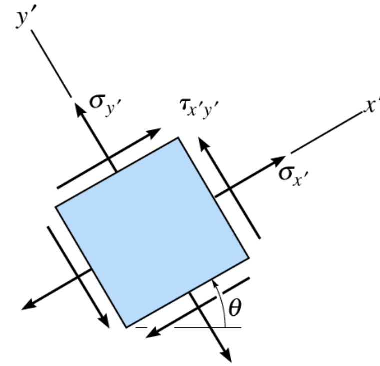
$$\sigma_{x'} \cdot \underline{\sin\theta} + \tau_{x'y'} \cdot \underline{\cos\theta} = \sigma_y \cdot \sin\theta + \tau_{xy} \cdot \cos\theta$$

For two-dimensional problems:



(a)





(b)

$$\begin{Bmatrix} \sigma_{x'} \\ \tau_{x'y'} \end{Bmatrix} = \begin{Bmatrix} \sigma_x \cdot \cos^2\theta + \tau_{xy} \sin\theta \cdot \cos\theta + \sigma_y \cdot \sin^2\theta + \tau_{xy} \sin\theta \cos\theta \\ -\sigma_x \cdot \sin\theta \cos\theta - \tau_{xy} \sin^2\theta + \sigma_y \cdot \sin\theta \cos\theta + \tau_{xy} \cdot \cos^2\theta \end{Bmatrix}$$

$$\boxed{\begin{aligned} \sigma_{x'} &= \sigma_x \cdot \cos^2\theta + \sigma_y \cdot \sin^2\theta + 2 \cdot \tau_{xy} \cdot \sin\theta \cdot \cos\theta \\ \tau_{x'y'} &= (\sigma_y - \sigma_x) \cdot \sin\theta \cdot \cos\theta + \tau_{xy} \cdot (\cos^2\theta - \sin^2\theta) \end{aligned}}$$

We use the following trigonometric relations…

$$\cos^2\theta = \frac{1+\cos(2\theta)}{2} \qquad \sin(2\theta) = 2\sin\theta\,\cos\theta$$

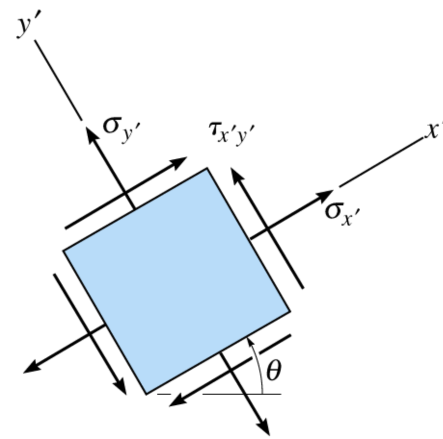
$$\sin^2\theta = \frac{1-\cos(2\theta)}{2} \qquad \cos(2\theta) = \cos^2\theta - \sin^2\theta$$

… to get $\qquad \sigma_{avg} = \dfrac{\sigma_x+\sigma_y}{2} \qquad \left(\dfrac{\sigma_x-\sigma_y}{2}\right)$

$$\sigma_{x'} = \left(\frac{\sigma_x+\sigma_y}{2}\right) + \left(\frac{\sigma_x-\sigma_y}{2}\right)^{*} \cos(2\theta) + \tau_{xy}\sin(2\theta)$$

$$\tau_{x'y'} = -\left(\frac{\sigma_x-\sigma_y}{2}\right)^{*} \sin(2\theta) + \tau_{xy}\cos(2\theta)$$

$$\sigma_{y'} = \left(\frac{\sigma_x+\sigma_y}{2}\right) - \left(\frac{\sigma_x-\sigma_y}{2}\right)^{*} \cos(2\theta) - \tau_{xy}\sin(2\theta)$$

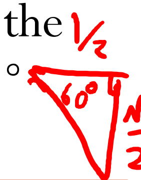


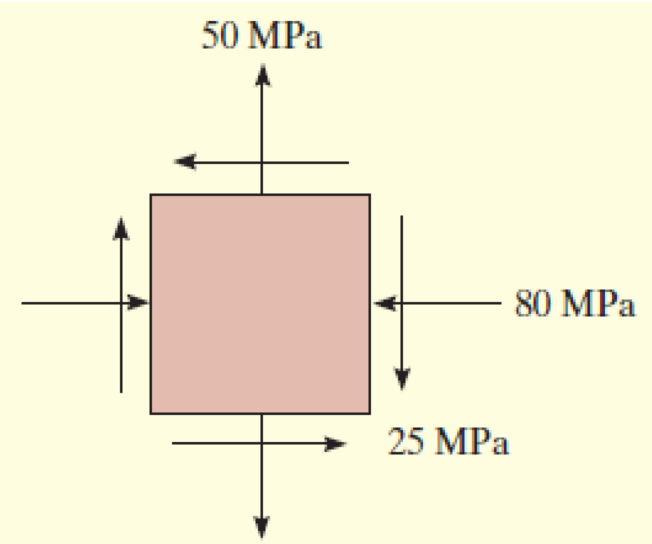
(b)

$$\frac{\sigma_{x'}+\sigma_{y'}}{2} = \sigma_{avg}$$

Note that: $\boxed{\dfrac{\sigma_{x'} + \sigma_{y'}}{2} = \dfrac{\sigma_x + \sigma_y}{2}} = \sigma_{avg}$

Example 1: The state of plane stress at a point is represented by the element shown in the figure below. Determine the state of stress at the point on another element oriented 30° clockwise from the position shown.





$2\theta = -60^\circ$

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \underbrace{\cos(2\theta)}_{1/2} + \tau_{xy} \sin(2\theta)$$

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2} \sin(2\theta) + \tau_{xy} \cos(2\theta)$$

$$\sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2} \cos(2\theta) - \tau_{xy} \sin(2\theta)$$

$$\underline{\sigma_{x'}} = \frac{-80+50}{2} + \frac{-80-50}{2}\left(\frac{1}{2}\right) + (-25)\cdot\left(-\frac{\sqrt{3}}{2}\right) = -25.85$$

MPa

$$\boxed{\sigma_{x'} = -25.85\,\text{MPa}}$$

$$\boxed{\tau_{x'y'} = -64.8\,\text{MPa}}$$

$$\boxed{\sigma_{y'} = -4.15\,\text{MPa}}$$