Moment Equilibrium

\[ M_z = 0 \Rightarrow -M - \int y \, dF = 0 \]
\[ -M \int y \, dF = \int F \, \sigma \, dA \]
\[ \sigma_x = \frac{E}{y} \int y \, dA \]
\[ \sigma_x = -\frac{M \cdot y}{I_z} \]

(b) Transverse section
\[ S_y dA = 0 \]
\[ E = \frac{F \cdot L}{E \cdot A} \]
\[ E = \frac{F \cdot L}{E \cdot A} \]
\[ I_z = \int y^2 \, dA \]

We want to relate \( M \) and \( \sigma_x \).
Use \[ \sigma_x = -\frac{E}{y} \int y \, dF = -\frac{\sigma_x}{y} \]
Centroid of an area

- The centroid of the area \( A \) is defined as the point \( C \) of coordinates \( \bar{x} \) and \( \bar{y} \), which satisfies the relation

\[
\int_A x \, dA = A \bar{x} \\
\int_A y \, dA = A \bar{y}
\]

- In the case of a composite area, we divide the area \( A \) into parts \( A_1, A_2, A_3 \)

\[
A_{total} \bar{x} = \sum A_i \bar{x}_i \\
A_{total} \bar{y} = \sum A_i \bar{y}_i
\]

Example: Find the centroid position in the \( yz \) coordinate system shown for \( t = 20 \text{cm} \)

\[
\bar{y} = \frac{A_1 \bar{y}_1 + A_2 \bar{y}_2}{A_1 + A_2} \\
\bar{y}_1 = \frac{z + t}{2} \\
\bar{y}_2 = \frac{z + 2t}{2} \\
A_1 = \frac{(2t)(3t)}{2} = 6t^2 \\
A_2 = 4t^2 \\
\bar{y} = \frac{(6t^2)(\frac{z + t}{2}) + (4t^2)(\frac{z + 2t}{2})}{6t^2 + 4t^2} \\
\bar{y} = \frac{(9 + 14)t}{10} = 2.3t
\]

\[
= 2.3 \times (20 \text{ cm}) = 46 \text{ cm}
\]
Second moment of area

- The 2nd moment of the area \( A \) with respect to the \( x \)-axis is given by
  \[ I_x = \int_A y^2 \, dA \]
- The 2nd moment of the area \( A \) with respect to the \( y \)-axis is given by
  \[ I_y = \int_A x^2 \, dA \]

- Example: 2nd moment of area for a rectangular cross section:
  
  \[ I_z = \int_{area} y^2 \, dA = \int_{0}^{h/2} \int_{0}^{w/2} y^2 \, dx \, dy = \frac{w^2}{3} \int_{0}^{h/2} dy = \frac{w^2}{3} \left( \frac{h}{2} \right) = \frac{w^2 h}{6} \]
  
  \[ = \frac{b}{3} \left( \left( \frac{h}{2} \right)^3 - \left( -\frac{h}{2} \right)^3 \right) = \frac{b}{3} \left( \frac{h^3}{8} + \frac{h^3}{8} \right) = \frac{b}{3} \cdot \frac{h^3}{4} \]
  
  \[ I_z = \frac{bh^3}{12} \text{ Valid for all rectangles!} \]

Centroids and area moments of area:

Formula sheet

<table>
<thead>
<tr>
<th>Moments and Geometric Centroids</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Q = \bar{y} \cdot A )</td>
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<tr>
<td>--------------------------------</td>
</tr>
<tr>
<td><strong>Rectangle</strong></td>
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<tr>
<td>( y )</td>
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<td></td>
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<tr>
<td><strong>Circle</strong></td>
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<tr>
<td>( y )</td>
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<td><strong>Semicircle</strong></td>
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<td>( y )</td>
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<tr>
<td><strong>Parallel Axis Theorem</strong></td>
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<td>( I_y = I_x + A \bar{d}_{yc}^2 )</td>
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</tbody>
</table>
Parallel-axis theorem: the 2nd moment of area about an axis through C parallel to the axis through the centroid C’ is given by

\[ I_C = I_{C'} + A d_C^2 \]

\[ I_{z'} = I_{z} + I_{dx} \]

Example: Find the 2nd moment of area about the horizontal axis passing through the centroid assuming \( t = 20 \) cm

\[ I_{a'z'} = \frac{4t^3}{12} + \left(4t^2\right)(1.2t)^2 \]

\[ A_1 = 4t^2 \]

\[ d_i = \bar{y}_1 - \bar{y} = 3.5 - 2.3 \]

\[ = 1.2t \]

\[ 1.2t^2 = 1.44t^2 \]

\[ 1.44 \times 4 = 5.76 \]

\[ \frac{1}{3} = 0.333... \]

\[ I_{z'} = 4t^2\left(\frac{1}{3} + 5.76\right) \]

\[ = 6.093 + 4 \]

\[ I_{2z'} = I_z + A_2 d_2^2 \]

\[ A_2 = 6 + \frac{(2t)(3.84)^3}{12} \]

\[ d_2 = \bar{y} - \bar{y}_a = 2.34 - 1.5t \]

\[ = 0.8t \]

\[ I_{2z'} = (2.25 + 3.84)4t^2 \]

\[ = 6.09 + 4 \]

\[ I_{z'} = 12.183 + 4 \]

Could also solve using this approach...
Bending stress formula

\[ \sigma_x(x, y) = -\frac{M(x) y}{I_z(x)} \]

- The maximum magnitude occurs the furthest distance away from the neutral axis. If we denote this maximum distance \( c \), consistent with the diagram below, then we can write:

\[ \sigma_m = \frac{|M|c}{I_y} \]

For positive bending moment \( M > 0 \):

- Compression \( \sigma_x < 0 \)
- Tension \( \sigma_x > 0 \)
**Example:** Find the maximum tensile and compressive stresses in this beam subjected to moment $M_x = 100 \text{ N-m}$ with the moment vector pointing in the direction of the z-axis. Again take $t = 30 \text{ cm}.$

\[ M_x = 100 \text{ N-m} \]

Compressive max $\sigma$ at top of beam
\[ \Rightarrow C = 4t - 2.3t = 1.7t \]

Compressive
\[ \sigma_{\text{max \ comp}} = \frac{|M| \cdot (1.7t)}{I_z} \]

Tensile max at bottom
\[ C = -2.3t \]
\[ \sigma_{\text{max \ tensile}} = \frac{|M| \cdot (2.3t)}{I_z} \]
Why I-beams?

- $I_z \uparrow$ with more material far from N.A.
- I beams nice because
  - rectangular plates easy to make
  - easy to create attachment
  - relatively inexpensive, give high $I_z$
  - can be extruded
  - high $I_z \Rightarrow$ low $\sigma_{\max}$

Summary of bending in beams

- Maximum stress due to bending
  \[ \sigma = \frac{Mc}{I} \]

- Bending stress is zero at the neutral axis and ramps up linearly with distance away from the neutral axis

- $I$ is the 2nd moment of area about the neutral axis of the cross section
  - Be sure to find the cross-section’s centroid and evaluate $I$ about an axis passing through the centroid, using the parallel axis theorem if needed
  \[ I_C = I_{C'} + A d_C^2 \]

- To determine stress sign, look at the internal bending moment direction:
  - Side that moment curls towards is in compression
  - Side that moment curls away from is in tension
Find $I_2$

$I = I_1 - 2I_2$

$I_1 = \frac{(6a)(5a)^3}{12} = \frac{625}{12} a^4$

$I_2 = \frac{a(3a)^3}{12} = \frac{27}{12} a^4$

$I = \left(\frac{625}{12} - 2 \left(\frac{27}{12}\right)\right) a^4$

$$I = \frac{625 - 54}{12} a^4 = \frac{571}{12} a^4$$

Find $I_2$.

1. Find $\bar{y}$

$$\bar{y} = \frac{A_1\bar{y}_1 + A_2\bar{y}_2}{A_1 + A_2}$$

$A_1 = (2r)(\frac{r}{2}) = r^2$

$\bar{y}_1 = r$

$A_2 = \frac{1}{2} \pi r^2$ from table

$\bar{y}_2 = 2r + \frac{4r}{3\pi}$

$$\bar{y} = \frac{r^3 + \left(\frac{1}{2} \pi r^2\right) \left(2r + \frac{4r}{3\pi}\right)}{r^2 + \frac{1}{2} \pi r^2}$$

$$= \left(1 + \frac{\pi r}{3} + \frac{2}{3} r\right) \left(\frac{6}{r}\right)$$

$$\bar{y} = \frac{(10 + 6\pi) r}{(6 + 3\pi)}$$

2. $I_2 = I_1 + I_2$

$$I_1 = \frac{(2r)(2r)^3}{12} = \frac{(r^2)^3}{12}$$

$$r^4 \left[ r - \frac{(10 + 6\pi) r}{6 + 3\pi} \right]^2$$
\[ I_1 = \frac{r^4}{12} + (r^2) \left( r - \frac{6 + 6\pi}{6 + 3\pi} \right) \]
\[ I_1 = \frac{r^4}{3} + r^2 \left( r - \frac{(10 + 6\pi)}{6 + 3\pi} \right)^2 \]

\[ I_2 = \left( \frac{\pi}{6} - \frac{9}{9\pi} \right) r^4 + A_\Delta \cdot d^2 \]

\[ A_\Delta = \frac{1}{3} \pi r^2 \]
\[ d = (2r + \frac{4r}{3\pi}) - \bar{y} \]
\[ = r \left( \frac{6\pi + 4}{3\pi} \right) - \left( \frac{10 + 6\pi}{6 + 3\pi} \right) r \]

Algebra...

\[ I_2 = I_1 + I_2 \]
\[ = \frac{r^4}{3} + r^4 \left( \frac{-4 - 3\pi}{6 + 3\pi} \right)^2 \]
\[ + \left( \frac{\pi}{6} - \frac{9}{9\pi} \right) r^4 \frac{\pi}{2} \left( r \cdot \left( \frac{6\pi + 4}{3\pi} \right) - \left( \frac{10 + 6\pi}{6 + 3\pi} \right) r \right)^2 \]

Messy, could be simplified further, but good enough to show the procedure.