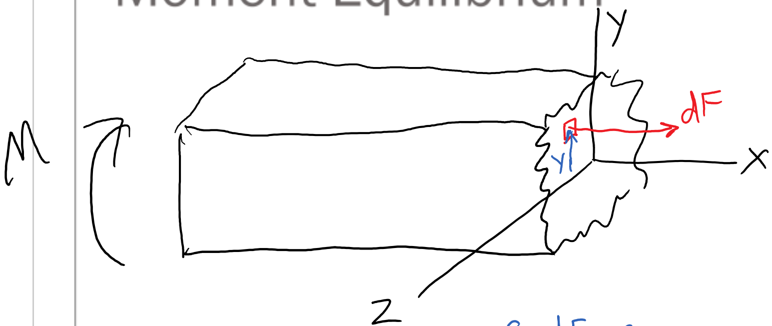
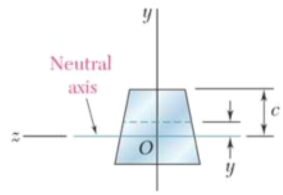
Moment Equilibrium





(b) Transverse section

$\Sigma M_z = 0 \Rightarrow -M - \int y \cdot dF = 0$

$-M = \int y \cdot dF$

$= \int y \cdot \left(\frac{-Ey}{\rho}\right) dA$

$= -\frac{E}{\rho} \int y^2 \cdot dA$

$dF = \sigma \cdot dA$

$\sigma_x = -\frac{Ey}{\rho}$

$\int y dA = 0$ is the first moment of area

$\int y^2 \cdot dA$ is the 2nd moment of area

bending moment

$\Rightarrow \int y^2 \cdot dA = \frac{M \cdot \rho}{E}$ curvature $\boxed{\int y^2 \cdot dA = I_z}$

material stiffness

$\underbrace{\boxed{\int y^2 \cdot dA = I_z}}_{\text{purely geometric Re: cross-sectional area}}$

load

$e = \frac{F \cdot L}{E \cdot A}$ length

matrl

geometric property

$$\frac{M \cdot \rho}{E} = \int_{area} y^2 dA = I_z \Rightarrow \boxed{M = \frac{E \cdot I_z}{\rho}}$$ — Moment-curvature equation

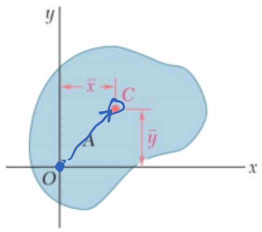
We want to relate M and σ_x

use $\sigma_x = -\frac{E \cdot y}{\rho} \Rightarrow \frac{E}{\rho} = -\frac{\sigma_x}{y} \Rightarrow \boxed{\sigma_x = \frac{-M \cdot y}{I_z}}$

Centroid of an area

- The centroid of the area A is defined as the point C of coordinates $\bar{x}$ and $\bar{y}$, which satisfies the relation

cross-sectional area



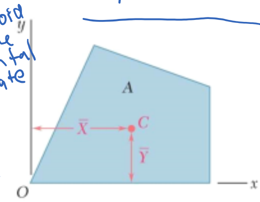
$$\boxed{\int_A x\, dA = A\bar{x}}$$

$$\boxed{\int_A y\, dA = A\bar{y}}$$

centroid in the horizontal coordinate

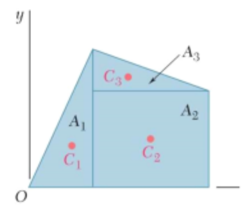
centroid in the vertical coordinate

$\int y \cdot dA = 0$ if $y = 0$ at the N.A. (centroid)



- In the case of a composite area, we divide the area A into parts A_1, A_2, A_3

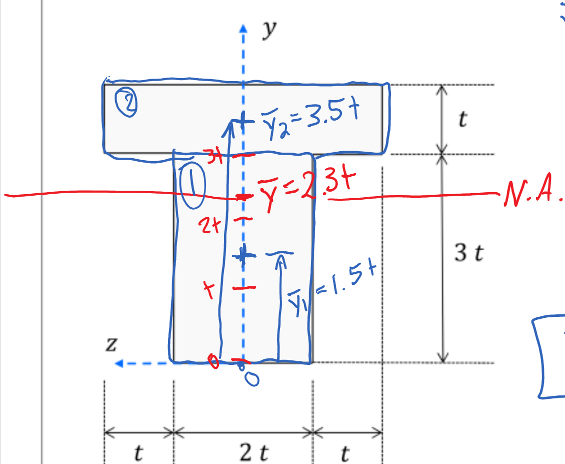
$=$



$$A_{total}\bar{x} = \sum_i A_i \bar{x}_i \qquad A_{total}\bar{y} = \sum_i A_i \bar{y}_i$$

$$\Rightarrow \bar{x} = \frac{\sum A_i \bar{x}_i}{A_{total}} \qquad \bar{y} = \frac{\sum_i A_i \bar{y}_i}{A_{total}} \qquad i = 1, 2, 3$$

Example: Find the centroid position in the yz coordinate system shown for $t = 20$cm



$$\bar{y} = \frac{A_1\bar{y}_1 + A_2\bar{y}_2}{A_1 + A_2}$$

$$\bar{y}_1 = \frac{3}{2}t \qquad \bar{y}_2 = \frac{7}{2}t$$

$$A_1 = (2t)(3t) = 6t^2 \qquad A_2 = 4t^2$$

$$\bar{y} = \frac{(6\cancel{t^2})(\frac{3}{2}t) + (4\cancel{t^2})(\frac{7}{2}t)}{6\cancel{t^2} + 4\cancel{t^2}}$$

$$\boxed{\bar{y} = \frac{(9 + 14)t}{10} = 2.3t}$$

$$= 2.3 \times (20\text{ cm})$$

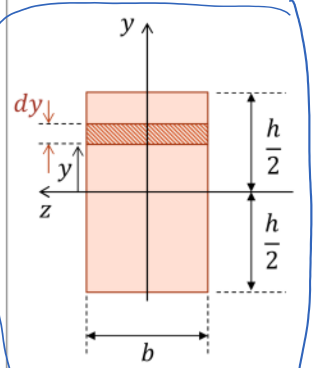
$$= 46\text{ cm}$$

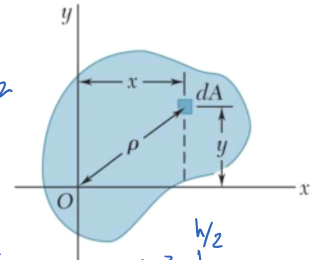
Second moment of area

- The 2nd moment of the area A with respect to the x-axis is given by $I_x = \int_A y^2 \boxed{dA}$ $=$ length^2

  $[y^2] = \text{length}^2$

  $[I_x] = \text{length}^4$
- The 2nd moment of the area A with respect to the y-axis is given by $I_y = \int_A x^2 \, dA$
- Example: 2nd moment of area for a rectangular cross section:





$I_z = \int_{area} y^2 \cdot dA \qquad dA = dy \cdot dz$

$= \int_{-h/2}^{h/2}\int_{-b/2}^{b/2} y^2 \cdot dz \cdot dy$

$= \int_{-b/2}^{b/2} dz \cdot \int_{-h/2}^{h/2} \cdot dy = b \cdot \int_{-h/2}^{h/2} y^2 dy = b\left(\frac{y^3}{3}\right)\Big|_{-h/2}^{h/2}$

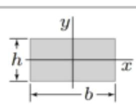
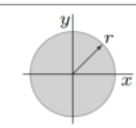
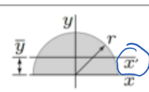
$= \frac{b}{3}\left[\left(\frac{h}{2}\right)^3 - \left(-\frac{h}{2}\right)^3\right] = \frac{b}{3}\left(\frac{h^3}{8} + \frac{h^3}{8}\right) = \frac{b}{3} \times \frac{h^3}{4}$

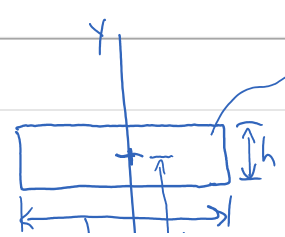
$\boxed{I_z = \frac{bh^3}{12}}$ Valid for all rectangles!

Centroids and area moments of area: Formula sheet

polar 2nd moment of area

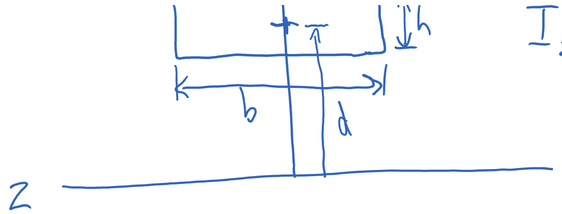
Moments and Geometric Centroids

	$Q = \bar{y}A$	$I_x = \int_A y^2 \, dA$	$J_o = \int_A \rho^2 \, dA$	$\bar{y} = \frac{1}{A}\int_A y \, dA$
Rectangle		$I_x = \frac{1}{12} bh^3$		
Circle		$I_x = \frac{\pi}{4} r^4$ $= \frac{\pi d^4}{64}$	$J_z = \frac{\pi}{2} r^4$ $= \frac{\pi d^4}{32}$	
Semicircle		$I_{x'} = \left(\frac{\pi}{8} - \frac{8}{9\pi}\right) r^4$		$\boxed{\bar{y} = \frac{4r}{3\pi}}$
Parallel Axis Theorem		$I_c = I_{c'} + A d_{cc'}^2$		



$I_z = \frac{bh^3}{12} + A \cdot d^2$

$= \frac{bh^3}{12} + b \cdot h \cdot d^2$



$$I_z = \frac{bh^3}{12} + A d^2$$
$$= \frac{bh^3}{12} + b \cdot h \cdot d^2$$

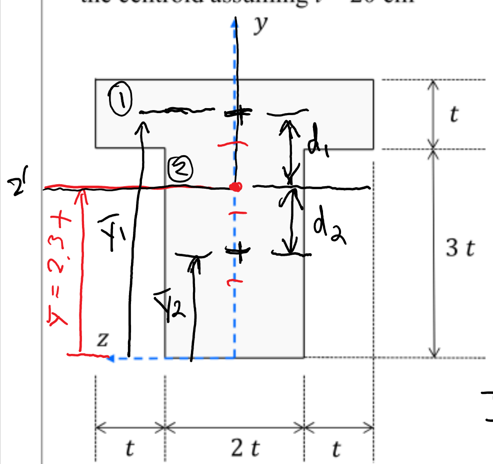
Parallel-axis theorem: the 2nd moment of area about an axis through C parallel to the axis through the centroid C' is given by

$$I_C = I_{C'} + A\, d^2_{CC'}$$

$\bar{y} = 2.3t$

$$I_{z'} = I_{1z'} + I_{2z'}$$

Example: Find the 2nd moment of area about the horizontal axis passing through the centroid assuming $t = 20$ cm



$$I_{1z'} = I_1 + A_1 \cdot d_1^2$$

$$I_1 = \frac{(4t)t^3}{12} \qquad A_1 = 4t^2 \qquad d_1 = \bar{y}_1 - \bar{y}$$
$$= 3.5t - 2.3t$$
$$= 1.2t$$

$$I_{1z'} = \frac{t^4}{3} + (4t^2)(1.2t)^2$$

$1.2t^2 = 1.44t^2$

$1.44 \times 4 = 5.76$

$\frac{1}{3} = 0.333\ldots$

$$\boxed{I_{1z'} = t^4\left(\frac{1}{3} + 5.76\right) = 6.09\bar{3}\,t^4}$$

$$I_{2z'} = I_2 + A_2 d_2^2 \qquad I_2 = \frac{(2t)(3t)^3}{12} = \frac{9}{2}t^4$$

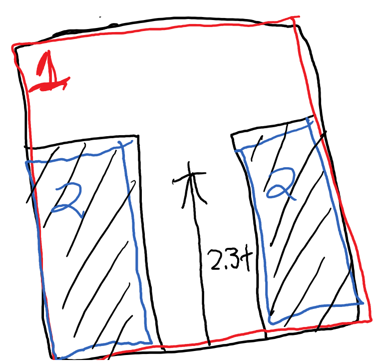
$$= \frac{9}{4}t^4 + 3.84t^4 \qquad A_2 = 6t^2$$

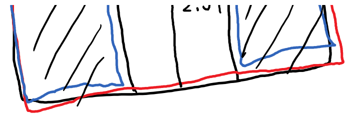
$$d_2 = \bar{y} - \bar{y}_2 = 2.3t - 1.5t$$
$$= 0.8t$$

$$I_{2z'} = (2.25 + 3.84)t^4 \qquad d_2^2 = 0.64t^2$$
$$= 6.09t^4$$

$$\boxed{I_{z'} = 12.18\bar{3}\,t^4}$$

Could also solve using this approach...





$$I = I_1 - 2 \cdot I_2$$

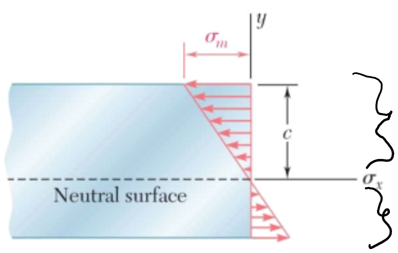
Bending stress formula

$$\sigma_x(x, y) = -\frac{M(x)\,y}{I_z(x)}$$

- The maximum magnitude occurs the furthest distance away from the neutral axis. If we denote this maximum distance "c", consistent with the diagram below, then we can write

$$\boxed{\sigma_{m_{max}} = \frac{|M|c}{I_z}}$$

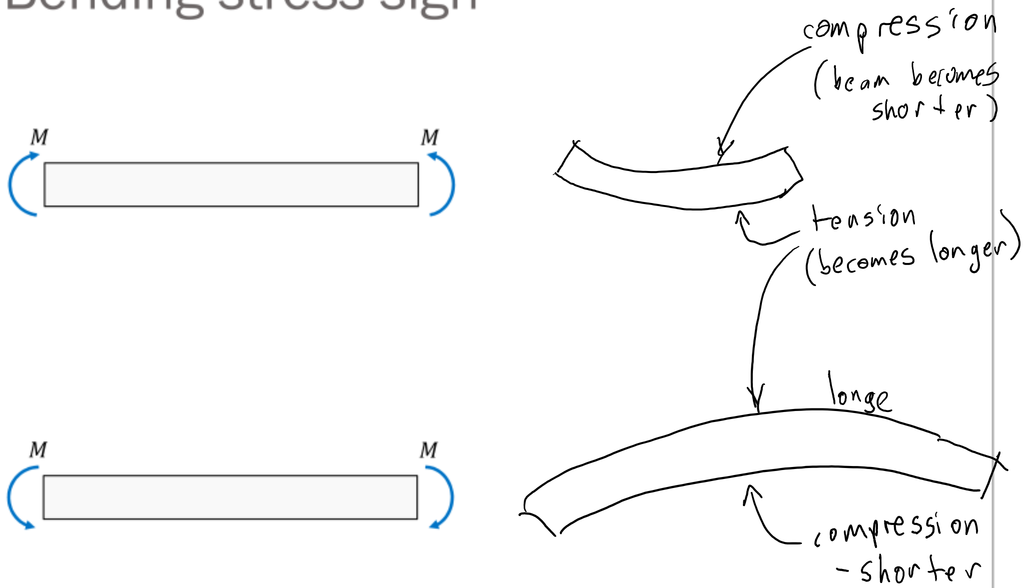
For positive bending moment $M > 0$



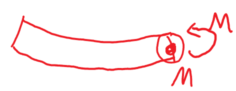
compression $\sigma_x < 0$

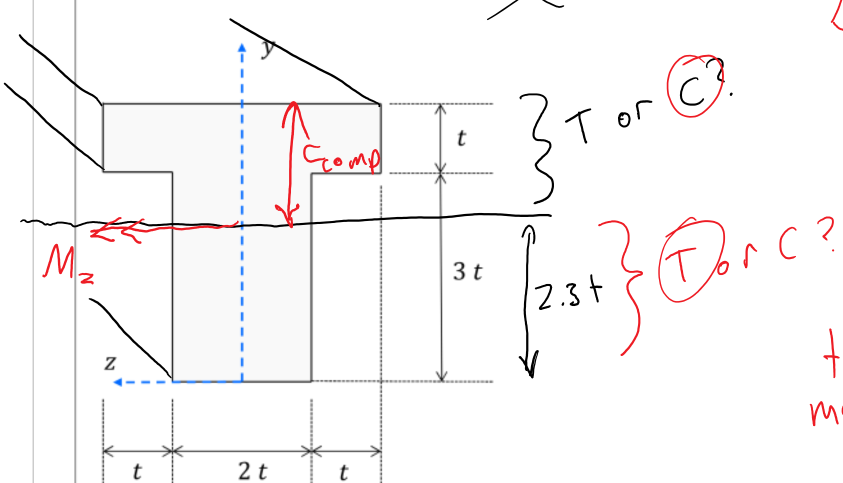
tension $\sigma_x > 0$

Bending stress sign



Example: Find the maximum tensile and compressive stresses in this beam subjected to moment $M_z = 100$ N-m with the moment vector pointing in the direction of the z-axis. Again take t = ~~20~~ cm.





compressive
max σ at
top of beam

$\Rightarrow c = 4t - 2.3t = 1.7t$

$$\sigma_{\substack{max \\ comp}} = \frac{|M| \cdot (1.7t)}{I_z}$$

tensile
max at bottom

$c = -2.3t$

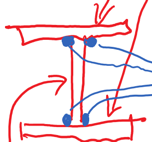
$$\sigma_{\substack{max \\ tensile}} = \frac{|M| \cdot (2.3t)}{I_z}$$

Why I-beams?

$I_z \uparrow$ with more material far from N.A.

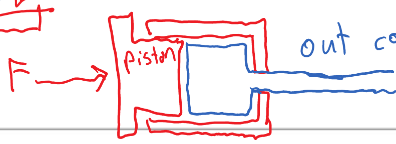
http://studio-tm.com/constructionblog/wp-content/uploads/2011/12/steel-i-beam-cantilevered-over-concrete-wall.jpg

I beams nice because
- rectangular plates easy to make
- easy to create attachment
- relatively inexpensive, give high I_z
- can be extruded



$$\sigma_x = \frac{-M \cdot y}{I_z}$$

high $I_z \Rightarrow$ low σ_{max}



out comes I-beam

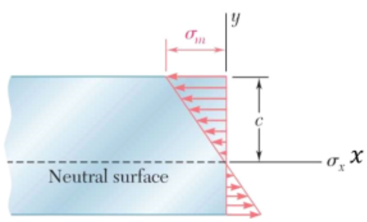
8020 material

Summary of bending in beams

- Maximum stress due to bending

$$\sigma = \frac{Mc}{I}$$

from $\sigma = \frac{-M \cdot y}{I}$

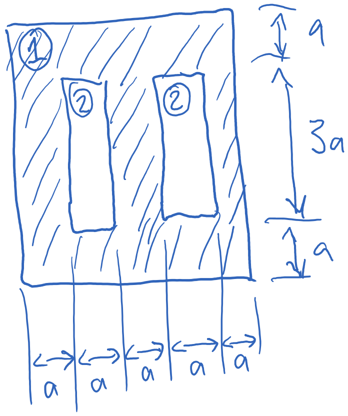


- Bending stress is zero at the neutral axis and ramps up linearly with distance away from the neutral axis
- I is the 2nd moment of area about the neutral axis of the cross section
  - Be sure to find the cross-section's centroid and evaluate I about an axis passing through the centroid, using the parallel axis theorem if needed

$$I_C = I_{C'} + A\, d_{CC'}^2$$

- To determine stress sign, look at the internal bending moment direction:
  - Side that moment curls **towards** is in **compression**
  - Side that moment curls **away from** is in **tension**

Find I_z



$$I = I_1 - 2I_2$$

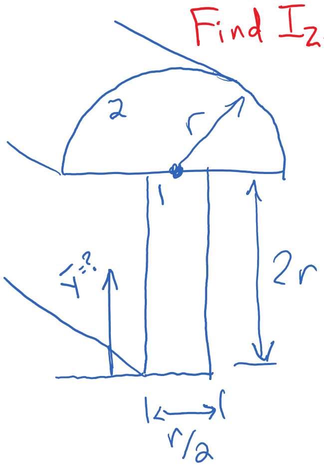
$$I_1 = \frac{(5a)(5a)^3}{12} = \frac{625}{12}a^4$$

$$I_2 = \frac{a \cdot (3a)^3}{12} = \frac{27}{12}a^4$$

$$I = \left[\frac{625}{12} - 2\left(\frac{27}{12}\right)\right]a^4$$

$$\boxed{I = \frac{625-54}{12}a^4 = \frac{571}{12}a^4}$$

Find I_z.



1. Find $\bar{y}$

$$\bar{y} = \frac{A_1\bar{y}_1 + A_2\bar{y}_2}{A_1 + A_2}$$

$$A_1 = (2r)\left(\frac{r}{2}\right) = r^2$$

$$\bar{y}_1 = r$$

$$A_2 = \frac{1}{2}\pi r^2 \quad \text{from table}$$

$$\bar{y}_2 = 2r + \frac{4r}{3\pi}$$

$$\bar{y} = \frac{r^3 + \left(\frac{1}{2}\pi r^2\right)\left(2r + \frac{4r}{3\pi}\right)}{r^2 + \frac{1}{2}\pi r^2}$$

$$= \left(\frac{r + \pi r + \frac{2}{3}r}{1 + \pi/2}\right)\left(\frac{6}{6}\right)$$

$$\bar{y} = \frac{(10 + 6\pi)r}{(6 + 3\pi)}$$

2. $I_z = I_1 + I_2$

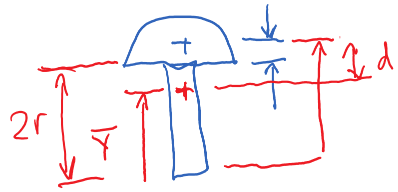
$$I_1 = \frac{(r/2)(2r)^3}{12} + (r^2)\left[r - \frac{(10+6\pi)r}{6+3\pi}\right]^2$$

$$I_1 = \frac{[\text{illegible}]}{12} + (r^2)\left[r - \frac{[\text{illegible}]}{6+3\pi}\right]$$

$$I_1 = \frac{r^4}{3} + r^2 \cdot \left[r - \frac{(10+6\pi)r}{6+3\pi}\right]^2$$

$$I_2 = \left(\frac{\pi}{8} - \frac{8}{9\pi}\right)r^4 + A_\triangle \cdot d^2$$

$$A_\triangle = \frac{1}{2}\pi r^2$$



$$d = \left(2r + \frac{4r}{3\pi}\right) - \bar{y}$$

$$= r\left(\frac{6\pi + 4}{3\pi}\right) - \left(\frac{10+6\pi}{6+3\pi}\right)r$$

Algebra...

$$I_z = I_1 + I_2$$

$$= \frac{r^4}{3} + r^4 \cdot \left(\frac{-4-3\pi}{6+3\pi}\right)^2$$

$$+ \left(\frac{\pi}{8} - \frac{8}{9\pi}\right)r^4 + \frac{\pi}{2}r^2 \cdot \left[r \cdot \left(\frac{6\pi+4}{3\pi}\right) - \left(\frac{10+6\pi}{6+3\pi}\right) \cdot r\right]^2$$

Messy, could be simplified further, but good enough to show the procedure.