Chapter 4: Axial Load

Chapter Objectives

✓ Determine the elastic deformation of axially loaded members
✓ Apply the principle of superposition for total effect of different loading cases
✓ Deal with compatibility conditions
✓ Deal with thermal stresses
✓ Misfit problems
Axial deformation

So far...

\[ \sigma = \frac{F}{A}, \quad \epsilon = \frac{\delta}{L}, \quad \sigma = E \epsilon \]

\[ \frac{F}{A} = E \frac{\delta}{L} \Rightarrow \delta = \frac{F \cdot L}{E \cdot A} \]

\[ F = k \cdot \Delta x \Rightarrow F = k \cdot \delta = \left( \frac{EA}{L} \right) \cdot \delta \]

In general, \( \epsilon = \frac{du}{dx} \) local material displacement

\[ F = E \cdot \frac{du}{dx} \Rightarrow du = \frac{F(x)}{E(x)A(x)} \cdot dx \]

\[ U(x_2) - U(x_1) = \int_{x_1}^{x_2} \frac{F(x)}{E(x)A(x)} \, dx \]
Superposition principle: 1) The loading must be linearly related to the stress or displacement that is to be determined; 2) Small deformation assumption

\[ \delta = \sum_i \delta_i = \sum_i \frac{F_i L_i}{E_i A_i} \]

\[ \delta_A = 0 \quad (A \text{ is fixed at wall}) \]

\[ \delta_{B/A} = \delta_B - \delta_A \quad \text{(Relative displacement of two points)} \]

\[ \delta_1 = \delta_B - \delta_A = \frac{F_1 L_1}{E_1 A_1} \]

\[ \delta_c/B = \delta_c - \delta_B = \delta_2 = \frac{F_2 L_2}{E_2 A_2} \]

\[ \delta_d/C = \delta_D - \delta_C = \delta_3 = \frac{F_3 L_3}{E_3 A_3} \]

Positive, small rods increase in length: \[ F_3 = F \]
\[ F_2 = F \]
\[ F_1 = F \]
Example 1

Rigid bar does not bend

Obtain the displacement of points C and D when the load P is applied.

Ground is fixed. Assume each bar has length L, E, all same.

Symmetric.

FBD:

\[ F_2 = P \text{ (tension)} \]

\[ F_1 = F_3 \text{ (symmetry)} \]

\[ \Sigma F_y = 0 \Rightarrow -F_1 - F_2 - F_3 = 0 \]

\[ F_1 = F_3 = -\frac{1}{2} F_2 = -\frac{P}{2} \text{ (compression)} \]
Example 1

Obtain the displacement of points $C$ and $D$ when the load $P$ is applied.

$F_1 - F_3 = -\frac{1}{2} P$  want $\delta_C$

$F_2 = P$

Use $\delta_{D/C} = \delta_D - \delta_C = \delta_2 = \frac{F_2 L}{EA}$

$$\delta_D = \delta_B = \delta_F \text{ because } \delta_A = \delta_E = 0$$

and green rod is rigid

$$\delta_{B/A} = \delta_B - \delta_A = \frac{F_1 L}{EA} - \frac{PL}{2EA} \Rightarrow \delta_B = \delta_D = \frac{-PL}{2EA}$$

$$\delta_C = \delta_D - \delta_{D/C} = \delta_D - \frac{F_2 L}{EA} = -\frac{PL}{2EA} - \frac{2PL}{2EA} = \frac{-3PL}{2EA}$$
Statically Determinate Problems

Can solve for all internal forces and reactions by using equilibrium alone.

\[ F_2 = P_C \]

\[ \sum F_x = 0 \Rightarrow F_2 = P_C \]

\[ F_1 - P_B + P_C = 0 \]

\[ F_1 = P_C - P_B \]

\[ R_A = P_B - P_C \]
Statically Indeterminate Problems

Motion is constrained at both A & C ⟹ Two reactions

\[ \Sigma F_x = 0 \text{ is the only equilibrium eqn. we can use} \]

\[ R_A - P_B + R_C = 0 \]

Apply compatibility condition ⟹ Geometric constraint.

If A & C are fixed, then \( \delta_1 + \delta_2 = 0 \)

\[ \frac{F_1 L_1}{E_1 A_1} + \frac{F_2 L_2}{E_2 A_2} = 0 \]

\[ |F_1| = |RA|, \ |F_2| = |RC| \]

Geometric constraint:

\[ F_2 \]

\[ R_C = F_2 \]

\[ F_1 = -R_A \]
Statically Indeterminate Problems

\[ F_2 = F_1 + P_B \]

\[ \frac{F_1 L_1}{E_1 A_1} + \frac{(F_1 + P_B) L_2}{E_2 A_2} = 0 \]

\[ \delta_1 + \delta_2 = 0 \]

\[ \frac{F_1 L_1}{E_1 A_1} = \frac{F_1 L_2}{E_2 A_2} = \frac{-P_B L_2}{E_2 A_2} \]

\[ F_1 \left( \frac{L_1}{E_1 A_1} + \frac{L_2}{E_2 A_2} \right) = \frac{-P_B L_2}{E_2 A_2} \]

\[ \ldots \text{can also solve for } F_2 \]
Given:
\[ A_1 = A_2 = A \]
\[ E_1 = E_2 = E \]
\[ h = 2b \]

Find:
\[ \sigma_1 = ? \]
\[ \sigma_2 = ? \]
\[ \delta_D = ? \]

Assume that BCD is rigid.

1. Equilibrium
\[ \Sigma M_B = 0 \]
\[ -\frac{W}{2} \cdot b - Wb + F_1 \cdot \sin \theta_1 \cdot b + F_2 \cdot \sin \theta_2 \cdot 2b = 0 \]
\[ F_1 \cdot \sin \theta_1 + 2 \cdot F_2 \cdot \sin \theta_2 = \frac{3}{2} W \]

2. Force-Elongation
\[ \epsilon_1 = \frac{F_1 \cdot l_1}{EA} \]
\[ \epsilon_2 = \frac{F_2 \cdot l_2}{EA} \]
Given:
\[ A_1 = A_2 = A \]
\[ E_1 = E_2 = E \]
\[ h = 2b \]

Find:
\[ \sigma_1 = ? \]
\[ \sigma_2 = ? \]
\[ \delta_D = ? \]

3. Geometric Compatibility (assume small rotation)

Similar triangles:
\[ \frac{\delta_c}{b} = \frac{\delta_D}{2b} \Rightarrow \delta_D = 2 \cdot \delta_c \] (4)

\[ \sin \theta_2 = \frac{e_2}{\delta_D} \] (5)

\[ \sin \theta_1 = \frac{e_1}{\delta_c} \] (6)

\[ \sin \theta_1 = \frac{2}{\sqrt{5}} \]
Given:

- $A_1 = A_2 = A$
- $E_1 = E_2 = E$
- $h = 2b$

Find:

- $\sigma_1 =$?
- $\sigma_2 =$?
- $\delta_D =$?

Apply $\delta_D = 2 \delta_C$

\[
\frac{2 F_2 b}{EA} = 2 \left( \frac{5 F_1 b}{2E_4} \right)
\]

\[
F_1 = \frac{2}{5} F_2
\]

Plug into (1) and solve

\[
\frac{2 F_2}{5} \frac{2}{\sqrt{5}} + 2 F_2 \frac{\sqrt{2}}{2} = \frac{3W}{2}
\]

\[
F_2 = \frac{75W}{8\sqrt{5} + 50\sqrt{2}}
\]

\[
F_1 = \frac{15W}{4\sqrt{5} + 25\sqrt{2}}
\]
Given:
\[ A_1 = A_2 = A \]
\[ E_1 = E_2 = E \]
\[ h = 2b \]

Find:
\[ \sigma_1 = ? \]
\[ \sigma_2 = ? \]
\[ \delta_D = ? \]

\[ \sigma_1 = \frac{F_1}{A} \]
\[ \sigma_2 = \frac{F_2}{A} \]

\[ \delta_D = \frac{2 \cdot F_2 \cdot b}{EA} \]
\[ = \frac{2b}{EA} \left( \frac{75 W}{8 \sqrt{5} + 25 \sqrt{2}} \right) \]
\[ = \left( \frac{75}{4 \sqrt{5} + 25 \sqrt{2}} \right) \frac{W \cdot b}{EA} \]
Find:
- Displacement
- Stress in the rod and tube

3. Geometric Compatibility

\[ e_r = e_t \]

\[ \frac{F_r}{E_1 A_1} = \frac{F_t}{E_2 A_2} \]

\[ F_r = F_t \cdot \left( \frac{E_1 A_1}{E_2 A_2} \right) \]

\[ F_t \cdot \left( \frac{E_1 A_1}{E_2 A_2} \right) + F_t \cdot \left( \frac{E_2 A_2}{E_2 A_2} \right) = -P \]

\[ F_t \cdot \left( \frac{E_1 A_1 + E_2 A_2}{E_2 A_2} \right) = -P \]

\[ F_t = \frac{-P \cdot E_2 A_2}{E_1 A_1 + E_2 A_2} \]
Find:
- Displacement
- Stress in the rod and tube

\[ F_r + F_t = -P \]
\[ F_r + \left( \frac{-P \cdot E_2 A_2}{E_1 A_1 + E_2 A_2} \right) = -P \cdot \left( \frac{E_1 A_1}{E_1 A_1 + E_2 A_2} \right) \]

\[ F_r = \frac{P}{E_1 A_1 + E_2 A_2} \cdot \left[ E_2 A_2 - E_1 A_1 - E_2 A_2 \right] \]
\[ F_r = -\frac{P \cdot E_1 A_1}{E_1 A_1 + E_2 A_2} \]

\[ e = e_t = e_r = \frac{F_t \cdot L}{E_2 A_2} \]
\[ e = \frac{-P \cdot L}{E_1 A_1 + E_2 A_2} \] Displaces to the left
Problems involving temperature changes

- Rod rests freely on a smooth horizontal surface
- Temperature of the rod is raised by $\Delta T$
- Rod elongates by an amount

$$\delta_T = \alpha L \Delta T$$

- $\alpha$: coefficient of thermal expansion ($/°C$)
- This deformation is associated to a thermal strain

$$\epsilon_T = \alpha \Delta T$$

NOTE: NO STRESS is associated with the thermal strain

Verrazano-Narrows Bridge: Because of thermal expansion of the steel cables, the bridge roadway is 12 feet (3.66 m) lower in summer than in winter
The device is used to measure a change in temperature. Rod AC and BD are made of Tungsten and Magnesium respectively. At a given temperature $T_0$, the rigid bar CDE is in the horizontal position. Determine an expression for the temperature $T$ as a function of the vertical displacement of point E, $\delta_E$.

- Rod AC: Tungsten $\alpha_t$
- Rod BD: Magnesium $\alpha_m$

\[ \alpha_m > \alpha_t \]

\[ DT = T - T_0 \]

\[ 2. \]

\[ \frac{e_m - e_t}{a} = \frac{\delta_E - e_t}{a + b} \]

\[ \delta_E = (e_m - e_t) \frac{a + b}{a} + e_t \]

A. Elongation
- Tungsten: $E_t = \alpha_t \cdot L \cdot DT$
- Magnesium: $E_m = \alpha_m \cdot L \cdot DT$
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- Rod AC: Tungsten $\alpha_t$
- Rod BD: Magnesium $\alpha_m$

$\alpha_m > \alpha_t$

\[ e_m = \alpha_m \Delta T \cdot L \]
\[ e_t = \alpha_t \Delta T \cdot L \]

\[ \delta_E = (\alpha_m \cdot \Delta T \cdot L - \alpha_t \cdot \Delta T \cdot L) \left( \frac{a+b}{a} \right) + \alpha_t \cdot \Delta T \cdot L \]

\[ \delta_E = \Delta T \cdot L \left[ (\alpha_m - \alpha_t) \left( \frac{a+b}{a} \right) + \alpha_t \right] \]

\[ T = T_o + \frac{\delta_E}{L \cdot \left( \frac{1}{\alpha_t} + (\alpha_m - \alpha_t) \left( \frac{a+b}{a} \right) \right)} \]
Measuring the coefficient of thermal expansion

http://www.youtube.com/watch?v=TDnLbjd429M
$\Delta T = 300^\circ C$
$\Delta \theta = 58^\circ$
Initially, rod of length $L$ is placed between two supports at a distance $L$ from each other.

- No internal forces $\rightarrow$ no stress or strain
- Equilibrium: $R_A = -R_B = 0$

$F = -R_A$  

Statically indeterminate problem!

- After raising the temperature, the total elongation of the rod is still zero!
- The total elongation is given by

$$\delta = \frac{FL}{EA} + \alpha L \Delta T = 0$$

$F = -\alpha EA \Delta T$

Rod is under compression

- The stress in the rod due to change in temperature is given by

$$\sigma = -\alpha E \Delta T$$
1. Equilibrium

\[ F_1 = F_2 \quad \text{and} \quad F_x = 0 \Rightarrow F_1 = F_2 = F \]

2. Force - elongation

\[ \sigma_1 = \frac{F_1 L}{EA_1} + \alpha \cdot \Delta T \cdot L \]
\[ \sigma_2 = \frac{F_2 L}{EA_2} \]


\[ \sigma_1 + \sigma_2 = 0 \Rightarrow \sigma_1 = -\sigma_2 \]

Assume perfect insulation at point B.

Find \( \sigma_1 \) & \( \sigma_2 \)

Find \( \delta_B \)

\[ \frac{F_1}{EA_1} + \alpha \Delta T A = -\frac{F_2}{EA_2} \]

\[ \frac{F}{E} \left( \frac{1}{A_1} + \frac{1}{A_2} \right) = -\alpha \cdot \Delta T \]

\[ \Rightarrow F \left( \frac{A_1 + A_2}{A_1 A_2} \right) = -\alpha \Delta T \]

\[ F = -\frac{\alpha \cdot \Delta T \cdot A_1 A_2 E}{A_1 + A_2} \]

\[ \sigma_1 = \frac{F}{A_1} = \ldots \quad \delta_B = \delta_1 \]

\[ \sigma_2 = \frac{F}{A_2} = \ldots \]
\[ E_1 = E_2 = E \]
\[ \alpha_1 = \alpha_2 = \alpha \]
\[ A_1, A_2 \]

\[ \delta_B = \delta_1 = \frac{F \cdot L}{EA_1} + \alpha \cdot \Delta T \cdot L \]

\[ F = -\alpha \cdot \Delta T \cdot A_1 \cdot A_2 \cdot E \]

\[ A_1 + A_2 \]

\[ \delta_B = -\alpha \cdot \Delta T \cdot A_2 \cdot L \]

\[ \frac{A_2}{A_1 + A_2} \]

\[ = \alpha \cdot \Delta T \cdot L \cdot \left( 1 - \frac{A_2}{A_1 + A_2} \right) \]

\[ = \alpha \cdot \Delta T \cdot L \cdot \left( \frac{A_1 + A_2 - A_2}{A_1 + A_2} \right) \]

\[ \delta_B = \frac{\alpha \cdot \Delta T \cdot L \cdot A_1}{A_1 + A_2} \]

B displaces to the right
Rods (1) and (2) have stiffnesses $k_1$ and $k_2$, respectively. At end A, the rod is connected to a rigid wall, but at end C there is a small gap $\delta$ between the original position of end C and the rigid wall. A force $P$ is applied at point B as shown.

a) What is the minimum required force $P_{\text{min}}$ required to close the gap between end C and the wall at the right?

b) Derive expressions for the stress in each bar if the applied force is $P > P_{\text{min}}$. Which bar(s) are in tension? Which are in compression?

$F_1 = k_1 \cdot e_1$

$F_2 = k_2 \cdot e_2$

To close the gap $e_2 = 0$, $e_1 = \delta$

$F_2 = 0$

$F_1 = P$

$\Rightarrow P = k_1 \cdot \delta$

$P_{\text{min}} = k_1 \cdot \delta$
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\[
\begin{align*}
F_1 &= P + F_2 \\
F_2 &= (\delta k_1 - P) k_2 \\
F_2 &= \frac{(\delta k_1 - P) k_2}{k_1 + k_2} \\
\sigma_2 &= \frac{F_2}{A_2} = \frac{(\delta k_1 - P) k_2}{(k_1 + k_2) A_2} \\
\text{see that } &\quad \delta k_1 = P_{\text{min}} \Rightarrow (P_{\text{min}} - P) < 0 \\
\Rightarrow &\quad \text{bar 2 is in compression!}
\end{align*}
\]

\[
\begin{align*}
F_1 &= P + F_2 \\
F_2 &= \frac{k_1 + k_2}{k_1 + k_2} + \frac{(\delta k_1 - P) k_2}{k_1 + k_2} \\
\sigma_1 &= \frac{F_1}{A_1} = \frac{(P + \delta k_1) k_1}{(k_1 + k_2) A_1} \\
\text{bar 1 is in tension!}
\end{align*}
\]
Two steel bars \( (E_s = 200 \text{ GPa} \text{ and } \alpha_s = 11.7 \times 10^{-6}/°C) \) are used to reinforce a brass bar \( (E_b = 105 \text{ GPa}, \alpha_b = 20.9 \times 10^{-6}/°C) \) that is subjected to a load \( P = 25 \text{ kN} \). When the steel bars were fabricated, the distance between the centers of the holes that were to fit on the pins was made 0.5 mm smaller than the 2 m needed. The steel bars were then placed in an oven to increase their length so that they would just fit on the pins. Following fabrication, the temperature in the steel bars dropped back to room temperature. Determine \((a)\) the increase in temperature that was required to fit the steel bars on the pins, \((b)\) the stress in the brass bar after the load is applied to it.

**a) Steel bars only**

\[
L = 2 \text{ m (desired length)} \\
L_{\text{fab}} = L - \delta_m = 2 \text{ m} - 0.0005 \text{ m} = 1.9995 \text{ m} \\
\delta_m \text{ (misfit)} \\
\text{Want } \delta_T = \alpha_s \cdot \Delta T \\
\Rightarrow \Delta T = \frac{\delta_m}{\alpha_s \cdot L_{\text{fab}}} \\
\Delta T = \frac{0.0005}{(11.7 \times 10^{-6}/°C)(1.9995 \text{ m})} = 21.4 °C
\]

**b) After steel is pinned to the brass:**

1) Steel cools to init. temp. (2) Steel & brass achieve a new length \( L' = 2 \text{ m} \) (3) Apply \( P \) to the system
Two steel bars \((E_s = 200 \text{ GPa} \text{ and } \alpha_s = 11.7 \times 10^{-6}/\text{°C})\) are used to reinforce a brass bar \((E_b = 105 \text{ GPa}, \alpha_b = 20.9 \times 10^{-6}/\text{°C})\) that is subjected to a load \(P = 25 \text{ kN}\). When the steel bars were fabricated, the distance between the centers of the holes that were to fit on the pins was made 0.5 mm smaller than the 2 m needed. The steel bars were then placed in an oven to increase their length so that they would just fit on the pins. Following fabrication, the temperature in the steel bars dropped back to room temperature.

Determine \((a)\) the increase in temperature that was required to fit the steel bars on the pins, \((b)\) the stress in the brass bar after the load is applied to it.