Chapter 2: Strain

Chapter Objectives

✓ Understand the concepts of normal and shear strain
✓ Apply the concept to determine the strain for various types of problems

Strain is a measure of geometric deformation.

Critical
become proficient
in symbolic algebra

Key: understand dimensional analysis

\[ [\sigma] = \frac{\text{force}}{\text{area}} \Rightarrow \text{stress} \]

\[ [\tau] = \frac{\text{force}}{\text{area}} \]

\[ [A] = \text{area} \]
A = \frac{\pi}{4} d^2 \quad [d] = \text{length}

\frac{\pi}{4} = 1 \quad [d^2] = \text{area} = \text{length}^2

\sigma = E \cdot \varepsilon

\hat{E} = \text{stress} \quad [\varepsilon] = \text{length}^0 = 1

Dimensions must match on both sides of any equation. \Rightarrow [\sigma] = [E \cdot \varepsilon] = [E] \cdot [\varepsilon]

**Question**

If a rectangular bar of some metal is heated uniformly, does its shape change?

\Rightarrow \text{presence of normal strain, but no shear strain}

**If a ring is held upright and stepped on, will its shape change?**

\Rightarrow \text{circular \rightarrow elliptic}

\Rightarrow \text{presence of shear strain (and normal strain)}

\text{due to normal strain}

\text{due to shear strain}

DEFORMATION: change in length or shape of a body when forces are applied (or change in temperature)
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Rubber membrane subject to tension

Rectangle of one aspect ratio to a rectangle of a different aspect ratio
**Extensional strain (normal strain)**

Change in length of a member divided by its original length (i.e., deformation per unit length)

\[ \varepsilon = \frac{\delta}{L} = \frac{L_{\text{final}} - L_{\text{initial}}}{L_{\text{initial}}} \]

\[ [\varepsilon] = 1 \]

Undeformed configuration

Deformed configuration

Uniform strain along member AB

**Strain is dimensionless!**

Recall point-wise definition of stress:

\[ \sigma = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A} \]

Similarly, we have a point-wise definition of strain:

\[ \varepsilon = \lim_{\Delta x \to 0} \frac{\Delta \delta}{\Delta x} = \frac{d\delta}{dx} \]
True vs Engineering Strain

We just defined "engineering strain", \( \varepsilon = \frac{\Delta L}{L_i} \), where the change in length is divided by the initial length.

"True strain" accounts for change in length of the bar as strain increases:

\[ \Delta \varepsilon_1 = \frac{\Delta L_1}{L_0} \]
\[ \varepsilon_{\text{true}} = \int_0^{L_f} d\varepsilon \]
\[ [d\varepsilon] = 1 \]
\[ \frac{d\varepsilon}{L} = 1 \]

\[ \Delta \varepsilon_2 = \frac{\Delta L_2}{L_1} \]
\[ = \ln \left( \frac{L_f}{L_0} \right) \]
\[ = \ln \left( \frac{L_0 + \delta}{L_0} \right) \]

For most practical engineering purposes (or "structural") the true strain is very, very, very small.

For those cases, \( \ln (1 + \varepsilon_{\text{eng.}}) \approx \varepsilon_{\text{eng.}} \).

\( \varepsilon_{\text{true}} \approx \varepsilon_{\text{eng.}} \)

Taylor expansion:

\[ \ln(1 + \varepsilon_{\text{eng}}) = \varepsilon_{\text{eng}} - \frac{1}{2} \varepsilon_{\text{eng}}^2 + \frac{1}{3} \varepsilon_{\text{eng}}^3 - \frac{1}{4} \varepsilon_{\text{eng}}^4 + \ldots \]

When \( \varepsilon_{\text{eng}} \ll 1 \), \( \varepsilon_{\text{eng,}}^2, \varepsilon_{\text{eng,}}^3 \), etc. = 0

(1 + \varepsilon_{\text{eng}}) = \varepsilon_{\text{eng}} \) for small \( \varepsilon_{\text{eng}} \)
# True vs Engineering Strain

For $L_i = 10^{''}$

<table>
<thead>
<tr>
<th>$\delta$</th>
<th>$\varepsilon_{\text{eng}} = \frac{\delta}{L_i}$</th>
<th>$\varepsilon_{\text{true}} = \ln\left(\frac{L_f}{L_i}\right)$</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01''</td>
<td>0.001</td>
<td>0.00099</td>
<td>0.05%</td>
</tr>
<tr>
<td>0.05''</td>
<td>0.005</td>
<td>0.00498</td>
<td>0.25%</td>
</tr>
<tr>
<td>0.1''</td>
<td>0.01</td>
<td>0.00995</td>
<td>0.5%</td>
</tr>
<tr>
<td>1''</td>
<td>0.1</td>
<td>0.0953</td>
<td>4.9%</td>
</tr>
<tr>
<td>5''</td>
<td>0.5</td>
<td>0.4054</td>
<td>23.3%</td>
</tr>
</tbody>
</table>

*Clearly acceptable for TAM251 analysis.*

*Usually in engineering, 5% error and less is acceptable.*
Example

Part of a control linkage of an airplane consists of a rigid member CDB and a flexible cable AB. If a force is applied at the end D of the member and causes a normal strain in the cable of 0.0035 mm/mm, determine the displacement of point D. Originally the cable is unstretched.

Method 1: Trigonometry

\[ E_{AB} = 0.0035 = \frac{L_{AB} - L'_{AB}}{L_{AB}} \]
\[ L_{AB} = 0.0035 L_{AB} \]
\[ L'_{AB} = L_{AB} (1 + 0.0035) = 501.75 \text{ mm} \]

**Law of cosines** to red triangle

\[ (501.75 \text{ mm})^2 = (300 \text{ mm})^2 + (400 \text{ mm})^2 - 2(300 \text{ mm})(400 \text{ mm})\cos (90^\circ + \theta) \]
\[ \therefore \theta = \cos^{-1}(\frac{625}{300}) \approx 0.419^\circ \]
\[ = 0.0073 \text{ rad} \]

\[ s_D^2 = (600 \text{ mm})^2 + (600 \text{ mm})^2 - 2(600 \text{ mm})^2 \cdot \cos(0.0073 \text{ rad}) \]
\[ s_D = 4.36 \text{ mm} \]
Part of a control linkage of an airplane consists of a rigid member CDB and a flexible cable AB. If a force is applied at the end D of the member and causes a normal strain in the cable of 0.0035 mm/mm, determine the displacement of point D. Originally the cable is unstretched.

**Method 2: Assume rotations are small**

\[ \delta_x = L \cdot \sin \theta \]
\[ \delta_y = L - L \cos \theta \]
For small \( \theta \)
\[ \sin \theta \approx \theta \]
\[ \cos \theta \approx 1 \]
\[ \tan \theta \approx \theta \]
\[ \delta_x \approx L \cdot \theta \]
\[ \delta_y \approx 0 \]

\[ \frac{\delta_{AB}}{L_{AB}} = 0.0035 \]
\[ \delta_{AB} = 1.75 \text{ mm} \]

\[ \begin{align*}
\delta_B & = \frac{\delta_B}{600 \text{ mm}} = \frac{\delta_B}{300 \text{ mm}} \\
\Rightarrow \delta_D & = 2 \cdot \delta_B
\end{align*} \]

**Extension of cable**
\[ \delta_{AB} = \delta_{B} \cdot \sin \alpha \]
\[ \delta_{B} = \frac{\delta_{AB}}{\sin \alpha} \]
\[ \delta_D = 2 \cdot \delta_B = \frac{2 \cdot \delta_{AB}}{4/\pi} = \frac{2 \cdot (1.75 \text{ mm})}{0.8} \]
\[ \delta_D = 4.375 \text{ mm} \]

Shear Strain
Shear Strain

Axial loads: change in length
Shear loads: change in angle/shape

Shear strain = Change in angle that was originally at 90 degrees \( \left( \frac{\pi}{2} \right) \)

\[ \gamma = \tan \gamma = \frac{\delta_y}{L} \approx \gamma \] for small \( \gamma \)

In General

\[ \gamma_A = \lambda + \beta = \frac{\delta_y}{L_y} + \frac{\delta_x}{L_x} \]

Example

The rectangular plate is deformed into the shape shown by the dashed lines.

Determine

a) the average normal strain along diagonal BD
b) the average shear strain at corner B

\[ \varepsilon_{BD} = \frac{L_{DB} - L_{BD}}{L_{BD}} \]

\[ L_{BD} = 500 \text{ mm} \]

\[ B' = (403, 2) \text{ mm} \]

\[ D' = (2, 302) \text{ mm} \]

\[ L_{B'D'} = \sqrt{(403 - 2)^2 + (2 - 302)^2} \text{ mm} \]

\[ \varepsilon_{BD} = \frac{500.6 - 500}{500} = 1.6 \times 10^{-3} \]

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\[ \gamma_B = \alpha + \beta \]

\[ \alpha = \tan^{-1}\left(\frac{2}{403}\right) = 0.0050 \]

\[ \beta = \tan^{-1}\left(\frac{2}{302}\right) = 0.0066 \]

\[ \therefore \gamma_B = 0.0050 + 0.0066 = 0.0116 \text{ rad} \]

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**Measurement of Strain**

- **Direct measurement:**
  - Initial and final lengths of some section of the specimen are measured, perhaps by some handheld device such as a ruler
  - Axial strain computed directly by following formula:
    \[ \epsilon = \frac{\delta}{L} = \frac{L_{\text{final}} - L_{\text{initial}}}{L_{\text{initial}}} \]
  - Accurate measurements of strain in this way may require a fairly large initial length
Measurement of Strain

- **Contact Extensometer:**
  - A clip-on device that can measure very small deformations
  - Two clips attach to a specimen before testing
  - The clips are attached to a transducer body
    \[
    \varepsilon = \frac{\delta}{L} = \frac{L_{\text{final}} - L_{\text{initial}}}{L_{\text{initial}}}
    \]
  - The transducer outputs a voltage
  - Changes in voltage output are converted to strain

Measurement of Strain

- Strain gages
  - Small electrical resistors whose resistance changes with strain
  - Change in resistance can be converted to strain measurement
  - Often sold as “rosettes,” which can measure normal strain in
    two or more directions
  - Can be bonded to test specimen
Measurement of Strain

- **Digital Image Correlation (DIC)**
  - Image placed on surface of test specimen
  - Image may consist of speckles or some regular pattern
  - Deformation of image tracked by digital camera
  - Image analysis used to determine multiple strain components

DIC system analyzing a notch fracture test, from trilin.com

Strain field in a notch fracture test, as measured using DIC. From barslab.lab.mcgill.ca