TAM 212. Midterm 2. Nov 7, 2013.
Discussion 'Quiz'

- There are 20 questions, each worth 5 points. (Quiz has just 5 questions.)
- You must not communicate with other students during this test.
- No electronic devices allowed.
- This is a 2 hour exam.
- Do not turn this page until instructed to do so.
- There are several different versions of this exam.

1. Fill in your information:

Full Name: ______________________________

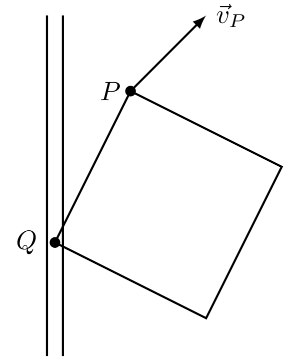
UIN (Student Number): ______________________________

NetID: ______________________________

2. Fill in the following answers on the Scantron form:

91. A
92. A
93. A
94. A
95. D
96. C

1. (1 point) A rigid body is moving in 2D as shown below with angular velocity $\vec{\omega} = \omega\hat{k}$. A pin at point Q constrains that point to move in a vertical slot.



Point P on the body has:

$$\vec{r}_{PQ} = -\hat{\imath} - 2\hat{\jmath} \text{ m}$$
$$\vec{v}_P = \hat{\imath} + \hat{\jmath} \text{ m/s.}$$

What is ω?

(A) ★ $-1 \text{ rad/s} \leq \omega < 0 \text{ rad/s}$

(B) $0 \text{ rad/s} < \omega < 1 \text{ rad/s}$

(C) $\omega = 0 \text{ rad/s}$

(D) $1 \text{ rad/s} \leq \omega$

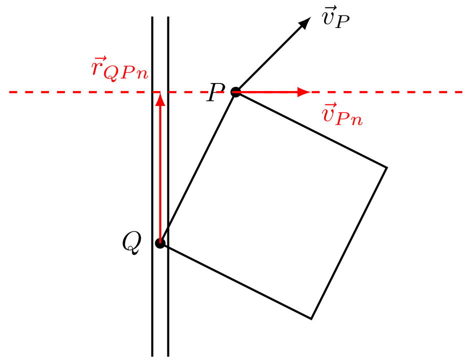
(E) $\omega < -1 \text{ rad/s}$

Solution. Taking $\vec{v}_Q = v_Q\,\hat{\jmath}$ with unknown speed v_Q, we have:

$$\begin{aligned}
\vec{v}_Q &= \vec{v}_P + \vec{\omega} \times \vec{r}_{PQ} \\
v_Q\,\hat{\jmath} &= \hat{\imath} + \hat{\jmath} + \omega\hat{k} \times (-\hat{\imath} - 2\hat{\jmath}) \\
&= \hat{\imath} + \hat{\jmath} + 2\omega\,\hat{\imath} - \omega\,\hat{\jmath} \\
&= (1 + 2\omega)\,\hat{\imath} + (1 - \omega)\,\hat{\jmath}
\end{aligned}$$

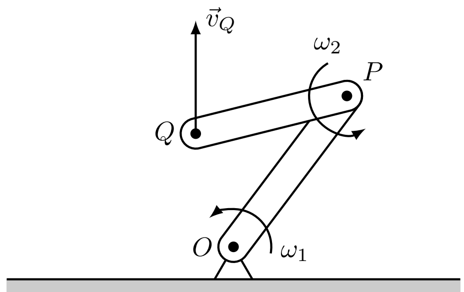
Equating $\hat{\imath}$ components gives $0 = 1 + 2\omega$, so $\omega = -0.5$ rad/s.

Alternatively, a fast solution is to observe that the rotation is caused by the normal velocity component $v_{Pn} = 1$ m/s acting at a distance $r_{PQt} = 2$ m, so the angular velocity is 1/2 rad/s clockwise.



---

2. (1 point) Two rods are connected with pin joints at O, P, and Q as shown. Rod OP has angular velocity $\vec{\omega}_1 = -\hat{k}$ rad/s and rod PQ has angular velocity $\vec{\omega}_2 = \omega_2 \hat{k}$.



The velocity $\vec{v}_Q$ of point Q is directly upwards and the positions of the rods are:

$$\vec{r}_{OP} = 3\hat{\imath} + 4\hat{\jmath} \text{ m}$$
$$\vec{r}_{PQ} = -4\hat{\imath} - \hat{\jmath} \text{ m}$$

What is the speed v_Q of point Q?

(A) $9 \text{ m/s} \le v_Q < 12 \text{ m/s}$

(B) $0 \text{ m/s} \le v_Q < 3 \text{ m/s}$

(C) $6 \text{ m/s} \le v_Q < 9 \text{ m/s}$

(D) $3 \text{ m/s} \le v_Q < 6 \text{ m/s}$

(E) ★ $12 \text{ m/s} \le v_Q$

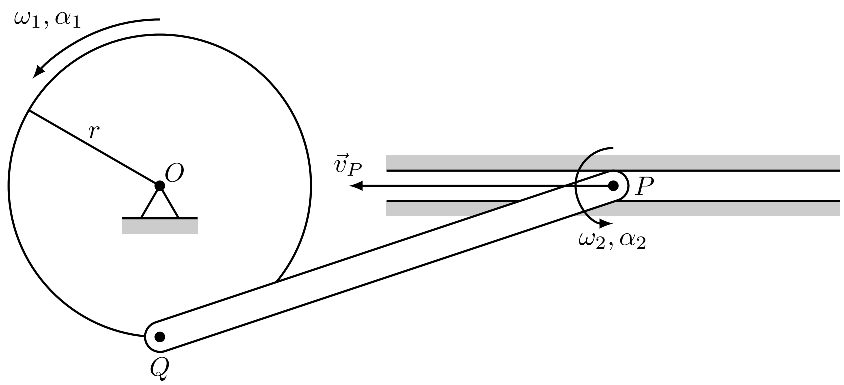
Solution. Starting from $\vec{v}_O = 0$ we have:

$$\begin{aligned} \vec{v}_P &= \vec{v}_O + \vec{\omega}_1 \times \vec{r}_{OP} \\ &= 4\hat{\imath} - 3\hat{\jmath} \\ \vec{v}_Q &= \vec{v}_P + \vec{\omega}_2 \times \vec{r}_{PQ} \\ &= (4 + \omega_2)\,\hat{\imath} + (-3 - 4\omega_2)\,\hat{\jmath} \text{ m/s.} \end{aligned}$$

Zero horizontal component implies $4 + \omega_2 = 0$, so $\omega_2 = -4$ rad/s. Then $\vec{v}_Q = 13\hat{\jmath}$ m/s, giving $v_Q = 13$ m/s.

---

3. (1 point) A circular rigid body with radius $r = 2$ m rotates about the fixed center O as shown. A rigid rod connects pins P and Q, and point P is constrained to only move horizontally. Point P has velocity $\vec{v}_P = -4\hat{\imath}$ m/s and acceleration $\vec{a}_P = 0$. The angular velocity and angular acceleration of the circular body are $\vec{\omega}_1 = \omega_1\hat{k}$ and $\vec{\alpha}_1 = \alpha_1\hat{k}$, while those of the rod are $\vec{\omega}_2 = \omega_2\hat{k}$ and $\vec{\alpha}_2 = \alpha_2\hat{k}$.



The position vectors are:

$$\vec{r}_{OQ} = -2\hat{\jmath} \text{ m}$$
$$\vec{r}_{PQ} = -6\hat{\imath} - 2\hat{\jmath} \text{ m}.$$

What is α_1?

(A) $1 \text{ rad/s}^2 \leq \alpha_1$

(B) $-1 \text{ rad/s}^2 \leq \alpha_1 < 0 \text{ rad/s}^2$

(C) $0 \text{ rad/s}^2 < \alpha_1 < 1 \text{ rad/s}^2$

(D) $\alpha_1 = 0 \text{ rad/s}^2$

(E) ★ $\alpha_1 < -1 \text{ rad/s}^2$

Solution. We first use the fact that $\vec{v}_Q = -v_Q\hat{\imath}$ to find:

$$\begin{aligned}\vec{v}_Q &= \vec{v}_P + \vec{\omega}_2 \times \vec{r}_{PQ} \\ -v_Q\hat{\imath} &= -4\hat{\imath} + \omega_2\hat{k} \times (-6\hat{\imath} - 2\hat{\jmath}) \\ (4 - v_Q)\hat{\imath} &= 2\omega_2\,\hat{\imath} - 6\omega_2\,\hat{\jmath}.\end{aligned}$$

Comparing $\hat{\jmath}$ components shows that $\omega_2 = 0$ and $v_Q = 4$ m/s, so $\omega_1 = -4/2 = -2$ rad/s. The fact that $\vec{v}_P = \vec{v}_Q$ could have also been realized from the fact that $\vec{v}_P$ and $\vec{v}_Q$ are parallel and offset, so there is no instantaneous center and thus no rotation.

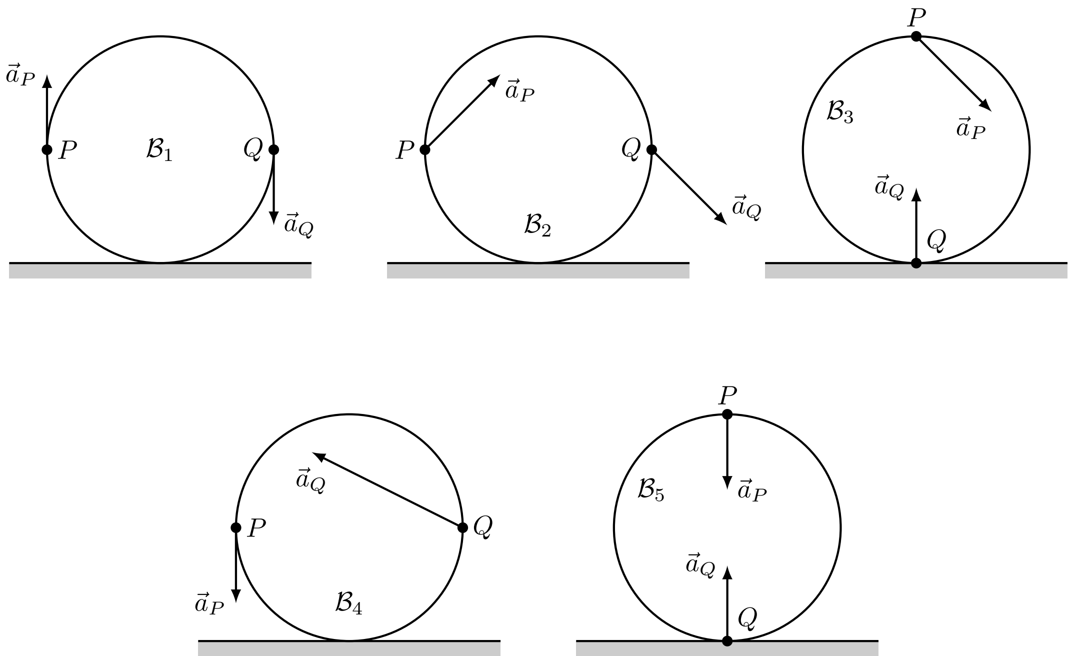
Now we have:

$$\begin{aligned}\vec{a}_Q &= \vec{a}_Q \\ \vec{a}_O + \vec{\alpha}_1 \times \vec{r}_{OQ} + \vec{\omega}_1 \times (\vec{\omega}_1 \times \vec{r}_{OQ}) &= \vec{a}_P + \vec{\alpha}_2 \times \vec{r}_{PQ} + \vec{\omega}_2 \times (\vec{\omega}_2 \times \vec{r}_{PQ}) \\ 0 + \alpha_1\hat{k} \times (-2\hat{\jmath}) - \omega_1^2(-2\hat{\jmath}) &= 0 + \alpha_2\hat{k} \times (-6\hat{\imath} - 2\hat{\jmath}) + 0 \\ 2\alpha_1\,\hat{\imath} + 2\omega_1^2\,\hat{\jmath} &= 2\alpha_2\,\hat{\imath} - 6\alpha_2\,\hat{\jmath}.\end{aligned}$$

Equating components shows that $\alpha_1 = \alpha_2 = -\omega_1^2/3 = -4/3 \approx -1.33 \text{ rad/s}^2$.

---

4. (1 point) Five circular rigid bodies are rolling without slipping as shown, with the accelerations of points P and Q on the bodies as drawn.



Which body does *not* have physically possible accelerations for points P and Q?

(A) $\mathcal{B}_5$

(B) $\mathcal{B}_2$

(C) $\mathcal{B}_3$

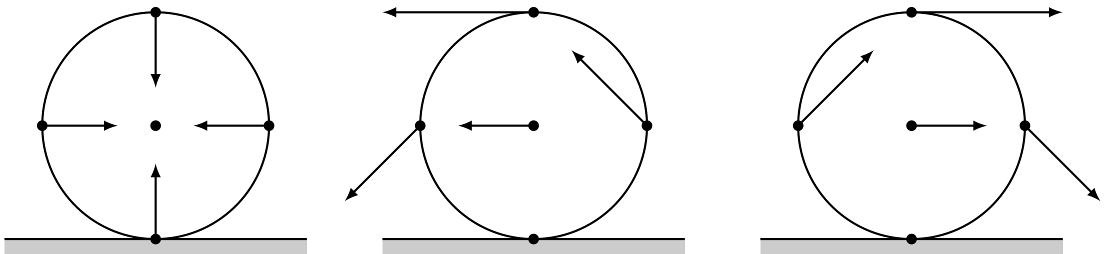
(D) ★ $\mathcal{B}_1$

(E) $\mathcal{B}_4$

Solution. The acceleration of a rolling body is determined by the angular velocity $\vec{\omega} = \omega\hat{k}$ and the angular acceleration $\vec{\alpha} = \alpha\hat{k}$. Taking C to be the center and P to be the contact point, any point Q on the body has acceleration:

$$\begin{aligned}
\vec{a}_Q &= \vec{a}_C + \vec{\alpha} \times \vec{r}_{CQ} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{CQ}) \\
&= \vec{\alpha} \times \vec{r}_{PC} + \vec{\alpha} \times \vec{r}_{CQ} - \omega^2 \vec{r}_{CQ} \\
&= \vec{\alpha} \times \vec{r}_{PQ} - \omega^2 \vec{r}_{CQ} \\
&= r\alpha \hat{r}_{PQ}^{\perp} - r\omega^2 \hat{r}_{CQ}.
\end{aligned}$$

Thus α causes rotational acceleration about the contact point, while ω causes centripetal acceleration towards the center, as shown below.

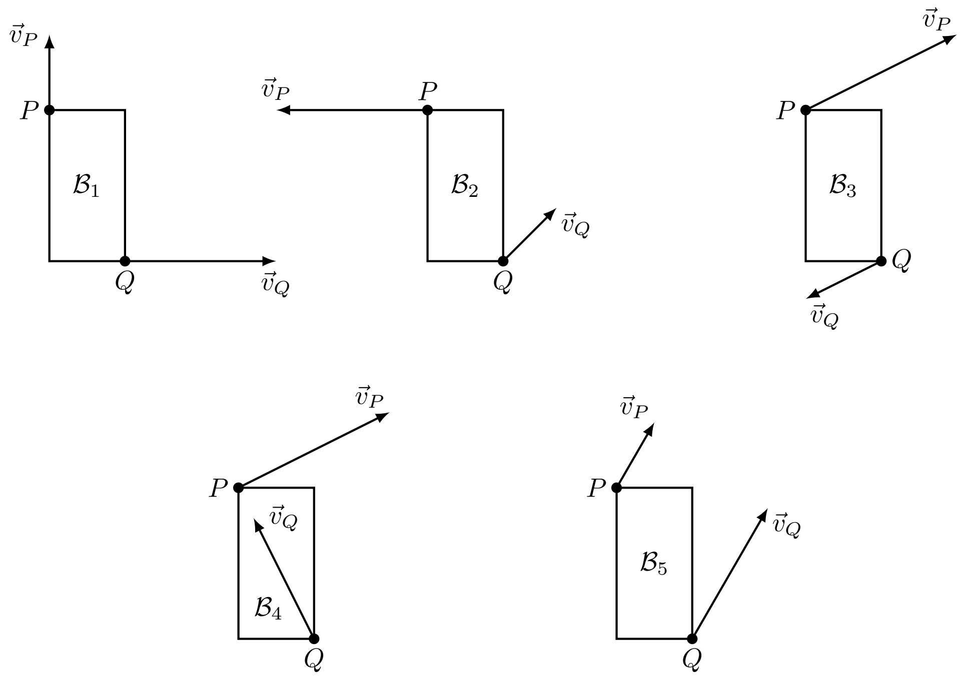


$\omega \neq 0$, $\alpha = 0$ $\qquad$ $\omega = 0$, $\alpha > 0$ $\qquad$ $\omega = 0$, $\alpha < 0$

Combining these, we see that:

- Body $\mathcal{B}_1$ is not possible.
- Body $\mathcal{B}_2$ has $\omega = 0$ and $\alpha < 0$, and it is possible.
- Body $\mathcal{B}_3$ has $\omega \neq 0$ and $\alpha < 0$ so that $2r|\alpha| = r\omega^2$, and it is possible.
- Body $\mathcal{B}_4$ has $\omega \neq 0$ and $\alpha > 0$ so that $r\alpha = r\omega^2$, and it is possible.
- Body $\mathcal{B}_5$ has $\omega \neq 0$ and $\alpha = 0$, and it is possible.

5. (1 point) Five bodies moving in 2D are shown below with the velocities of points P and Q on the bodies as drawn.



Which body has physically possible velocities for point P and Q?

(A) $\mathcal{B}_5$

(B) ★ $\mathcal{B}_3$

(C) $\mathcal{B}_2$

(D) $\mathcal{B}_4$

(E) $\mathcal{B}_1$

Solution.

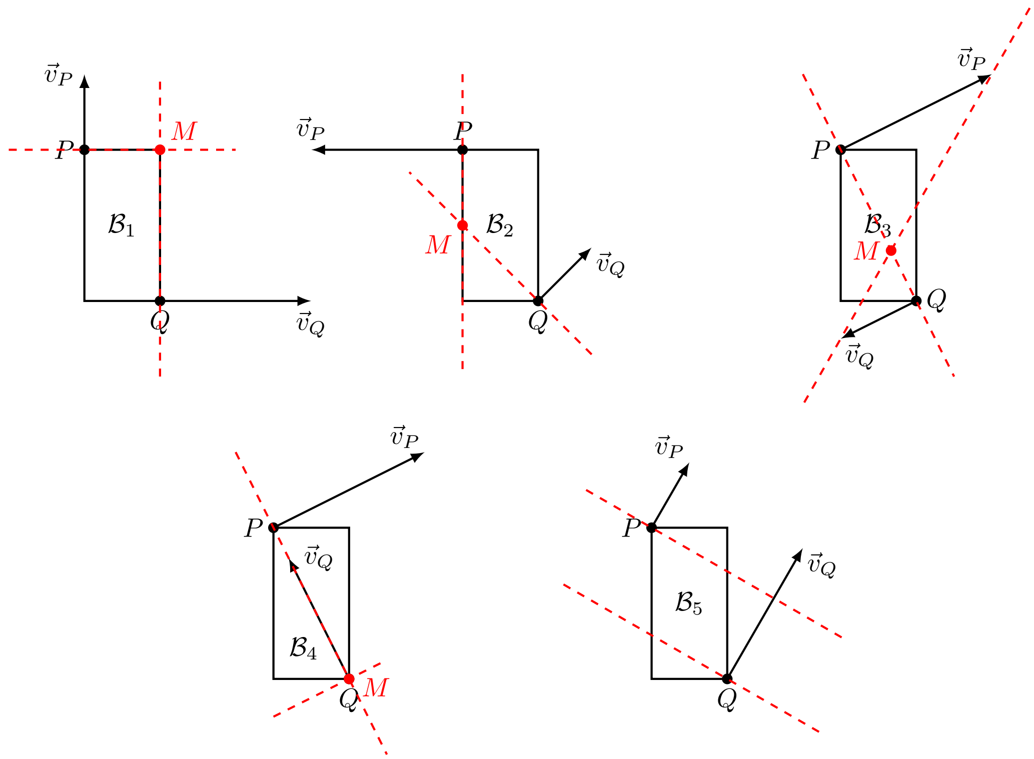
- Body $\mathcal{B}_1$ has P rotating clockwise and Q rotating counterclockwise about M, so it is impossible.

- Body $\mathcal{B}_2$ has P closer to M than Q, but v_P is larger than v_Q, so it is impossible.

- Body $\mathcal{B}_3$ is possible.

- Body $\mathcal{B}_4$ has M at Q, but $\vec{v}_Q \neq 0$, so it is impossible.

- Body $\mathcal{B}_5$ does not have an instantaneous center, and the velocities of P and Q are not equal, so it is impossible.



$\vec{v}_P$
P
M
$\mathcal{B}_1$
Q
$\vec{v}_Q$
$\vec{v}_P$
P
M
$\mathcal{B}_2$
$\vec{v}_Q$
Q
$\vec{v}_P$
P
$\mathcal{B}_3$
M
Q
$\vec{v}_Q$
$\vec{v}_P$
P
$\vec{v}_Q$
$\mathcal{B}_4$
Q
M
$\vec{v}_P$
P
$\vec{v}_Q$
$\mathcal{B}_5$
Q