TAM 212. Final. Dec 19, 2013.
'Quiz'

- There are 30 questions, each worth 5 points. (Quiz has just 5 questions.)
- You must not communicate with other students during this test.
- No electronic devices allowed.
- This is a 3 hour exam.
- Do not turn this page until instructed to do so.
- There are several different versions of this exam.

1. Fill in your information:

Full Name: ______________________________

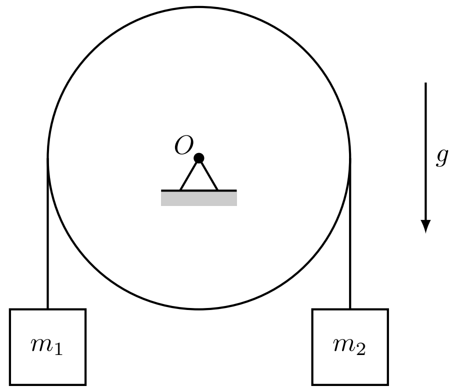
UIN (Student Number): ______________________________

NetID: ______________________________

2. Fill in the following answers on the Scantron form:

91. A
92. A
93. A
94. A
95. D
96. C

1. (5 points) A rigid wheel with radius r and moment of inertia I_O is pinned at point O. An inextensible massless rope connects two masses m_1 and m_2, and moves without slipping on the wheel. Gravity g acts downwards.



At the instant shown, all bodies are stationary and we have:

$$r = 2 \text{ m}$$
$$I_O = 16 \text{ kg m}^2$$
$$m_1 = 2 \text{ kg}$$
$$m_2 = 4 \text{ kg}$$
$$g = 10 \text{ m/s}^2$$

What is the magnitude of the angular acceleration $\vec{\alpha}$ of the wheel?

(A) $0 \text{ rad/s}^2 < \alpha < 1 \text{ rad/s}^2$

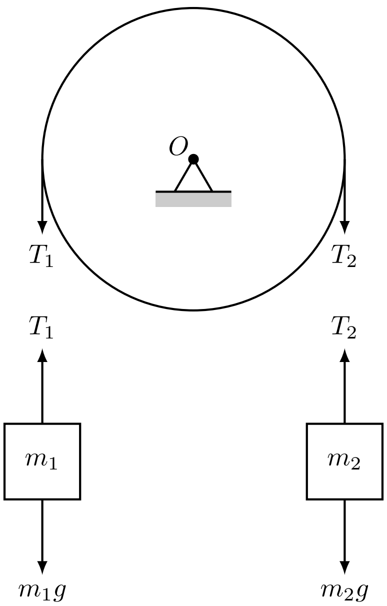
(B) $2 \text{ rad/s}^2 \leq \alpha < 3 \text{ rad/s}^2$

(C) ★ $1 \text{ rad/s}^2 \leq \alpha < 2 \text{ rad/s}^2$

(D) $3 \text{ rad/s}^2 \leq \alpha$

(E) $\alpha = 0 \text{ rad/s}^2$

Solution. Taking $\vec{\alpha} = \alpha\hat{k}$, we have that the acceleration of mass m_1 is $\vec{a}_1 = -r\alpha\hat{\jmath}$ and that of mass m_2 is $\vec{a}_2 = r\alpha\hat{\jmath}$. The free body diagram is:



Newton's equations for each mass and Euler's equations for the wheel give:

$$\left.\begin{aligned} T_1\hat{\jmath} - m_1 g\hat{\jmath} = m_1\vec{a}_1 = -m_1 r\alpha\hat{\jmath} \\ T_2\hat{\jmath} - m_2 g\hat{\jmath} = m_2\vec{a}_2 = m_2 r\alpha\hat{\jmath} \\ T_1 r\hat{k} - T_2 r\hat{k} = I_O\vec{\alpha} = I_O\alpha\hat{k} \end{aligned}\right\} \implies \begin{cases} \alpha = -1\ \text{rad/s}^2 \\ T_1 = 24\ \text{N} \\ T_2 = 32\ \text{N} \end{cases}$$

The magnitude of the acceleration is thus $\alpha = 1\ \text{rad/s}^2$.

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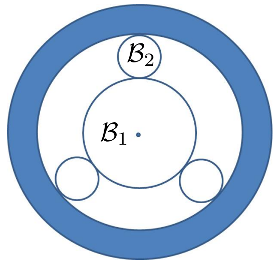
2. (5 points) A bearing is depicted below in which four discs roll without slipping inside a circular cavity which does *not* move.



At all times the body $\mathcal{B}_1$ is centered in the cavity, and at the time shown body $\mathcal{B}_2$ is directly above $\mathcal{B}_1$. The radius of $\mathcal{B}_1$ is 20 cm and the radius of $\mathcal{B}_2$ is 5 cm. If the angular velocity and angular acceleration of $\mathcal{B}_1$ are $\vec{\omega}_1 = 10\hat{k}$ rad/s and $\vec{\alpha}_1 = -\hat{k}$ rad/s^2, what is the angular acceleration $\vec{\alpha}_2$ of body $\mathcal{B}_2$?

(A) $\vec{\alpha}_2 = -4\hat{k}$ rad/sec^2.

(B) ★ $\vec{\alpha}_2 = 2\hat{k}$ rad/sec^2.

(C) $\vec{\alpha}_2 = 4\hat{k}$ rad/sec^2.

(D) $\vec{\alpha}_2 = -2\hat{k}$ rad/sec^2.

(E) $\vec{\alpha}_2 = 8\hat{k}$ rad/sec^2.

Solution. Let A_1 and A_2 be the points at which $\mathcal{B}_1$ and $\mathcal{B}_2$ are in contact. We know that $\vec{v}_{A_1} = \vec{v}_{A_2}$ and so $\vec{\omega}_1 \times (20\hat{\jmath}) = \vec{\omega}_2 \times (-10\hat{\jmath})$, and so $\omega_2 = -2\omega_1$. This relationship holds at all times and so we have $\dot{\omega}_2 = -2\dot{\omega}_1$; namely, at the time shown $\alpha_2 = (-2)(-1) = 2$ rad/s^2.

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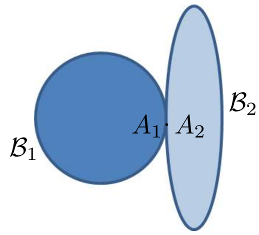
3. (5 points) Two rigid bodies bodies are in contact, rolling without slipping as seen below.



The point of contact on body $\mathcal{B}_1$ is A_1 and the point of contact on body $\mathcal{B}_2$ is A_2. In the configuration shown, which could be the acceleration vectors of these two points?

(A) $\vec{a}_{A_1} = -\hat{\jmath}$ m/s^2 and $\vec{a}_{A_2} = \hat{\jmath}$ m/s^2.

(B) $\vec{a}_{A_1} = -2\hat{\imath} + 9\hat{\jmath}$ m/s^2 and $\vec{a}_{A_2} = -2\hat{\imath} + 11\hat{\jmath}$ m/s^2.

(C) $\vec{a}_{A_1} = 5\hat{\imath} + 5\hat{\jmath}$ m/s^2 and $\vec{a}_{A_2} = -5\hat{\imath} - 5\hat{\jmath}$ m/s^2.

(D) $\vec{a}_{A_1} = -1\hat{\imath} + 10\hat{\jmath}$ m/s^2 and $\vec{a}_{A_2} = -\hat{\imath} - 10\hat{\jmath}$ m/s^2.

(E) ★ $\vec{a}_{A_1} = -\hat{\imath} + \hat{\jmath}$ m/s^2 and $\vec{a}_{A_2} = \hat{\jmath}$ m/s^2.

Solution. As discussed in lecture, the no slip condition implies that the tangential acceleration of the contact points must be equal; the normal components can be different. In the configuration given the tangential direction is given by $\hat{\jmath}$, so $\vec{a}_{A_1}$ and $\vec{a}_{A_2}$ must have the same $\hat{\jmath}$ component — there is only one such option.

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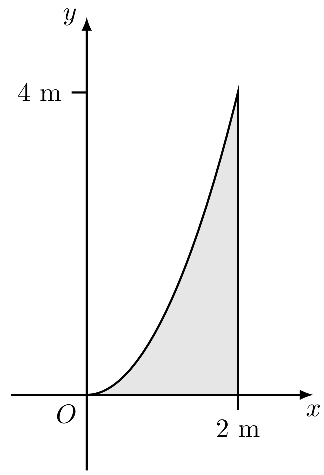
4. (5 points) A body has uniform thickness in the z direction and uniform density, and its shape in the x–y plane is bounded by the curves $y = x^2/\text{m}$, $y = 0$ m, and $x = 2$ m, as shown below.



What is the x coordinate C_x of the center of mass C of the body?

(A) $1.8 \text{ m} \leq C_x$

(B) ★ $1.5 \text{ m} \leq C_x < 1.6 \text{ m}$

(C) $1.6 \text{ m} \leq C_x < 1.7 \text{ m}$

(D) $C_x < 1.5 \text{ m}$

(E) $1.7 \text{ m} \leq C_x < 1.8 \text{ m}$

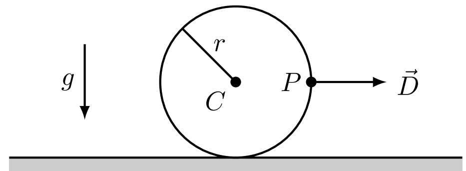
Solution. For thickness h and density ρ, the total mass is

$$m = \int_{0 \text{ m}}^{2 \text{ m}} \rho h (x^2/\text{m})\, dx$$
$$= \frac{8}{3} \rho h \text{ m}^2.$$

The x coordinate of the center of mass is then:

$$C_x = \frac{1}{m} \int_0^2 \rho h x (x^2/\text{m})\, dx$$
$$= \frac{1}{\frac{8}{3}\rho h \text{ m}^2} 4 \rho h \text{ m}^3$$
$$= 1.5 \text{ m}.$$

5. (5 points) A circular rigid body with center of mass C, mass $m = 2$ kg, moment of inertia $I_C = 1$ kg m^2, and radius $r = 1$ m is sitting on the ground as shown. The coefficient of friction between the body and the ground is $\mu = 0.1$. A driving force $\vec{D} = 3\hat{\imath}$ N acts at point P, and gravity $g = 10$ m/s^2 acts vertically.



What is the magnitude of the friction force $\vec{F}$?

(A) $F = 4$ N

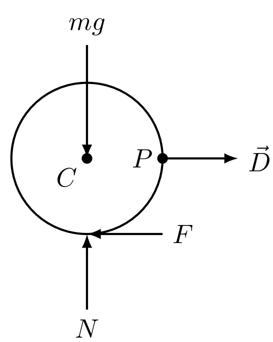
(B) $F = 2$ N

(C) ★ $F = 1$ N

(D) $F = 3$ N

(E) $F = 0$ N

Solution. With friction $\vec{F} = -F\hat{\imath}$ and normal force $\vec{N} = N\hat{\jmath}$, the free body diagram is:



Assuming sticking and taking $\vec{a} = a\hat{\imath}$ and $\vec{\alpha} = \alpha\hat{k}$, we have:

$$\left.\begin{aligned} \vec{D} - F\hat{\imath} + N\hat{\jmath} - mg\hat{\jmath} &= m\vec{a} = ma\hat{\imath} \\ -Fr\hat{k} &= I_C\vec{\alpha} = I_C\alpha\hat{k} \\ a &= -r\alpha \end{aligned}\right\} \Longrightarrow \begin{cases} N = 20 \text{ N} \\ F = 1 \text{ N} \\ a = 1 \text{ m/s}^2 \\ \alpha = -1 \text{ rad/s}^2 \end{cases}$$

Checking the Coulomb friction condition gives:

$$\begin{aligned} |F| &\stackrel{?}{\leq} \mu|N| \\ 1 &\stackrel{?}{\leq} 0.1 \times 20 \\ 1 &\leq 2 \end{aligned}$$

Thus it is sticking and $F = 1$ N.
