Recap

Derivative of a vector:

$$\dot{\vec{v}} = \frac{d}{dt}\vec{v}(t) = \lim_{\Delta t \to 0} \frac{\vec{v}(t+\Delta t) - \vec{v}(t)}{\Delta t}$$

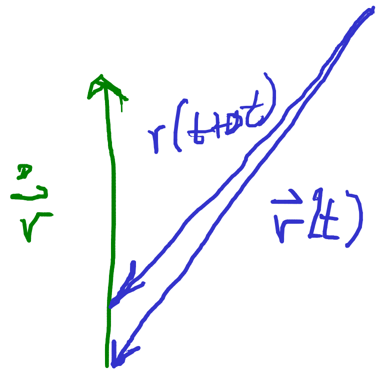
In Cartesian coordinates we differentiate components:

$$\vec{v} = x\hat{\imath} + y\hat{\jmath}$$

$$\dot{\vec{v}} = \dot{x}\,\hat{\imath} + \dot{y}\,\hat{\jmath}$$

The derivative $\dot{\vec{v}}$ "pulls along" $\vec{v}$ graphically

Ex



At the instant shown, $\dot{\vec{r}}$ is

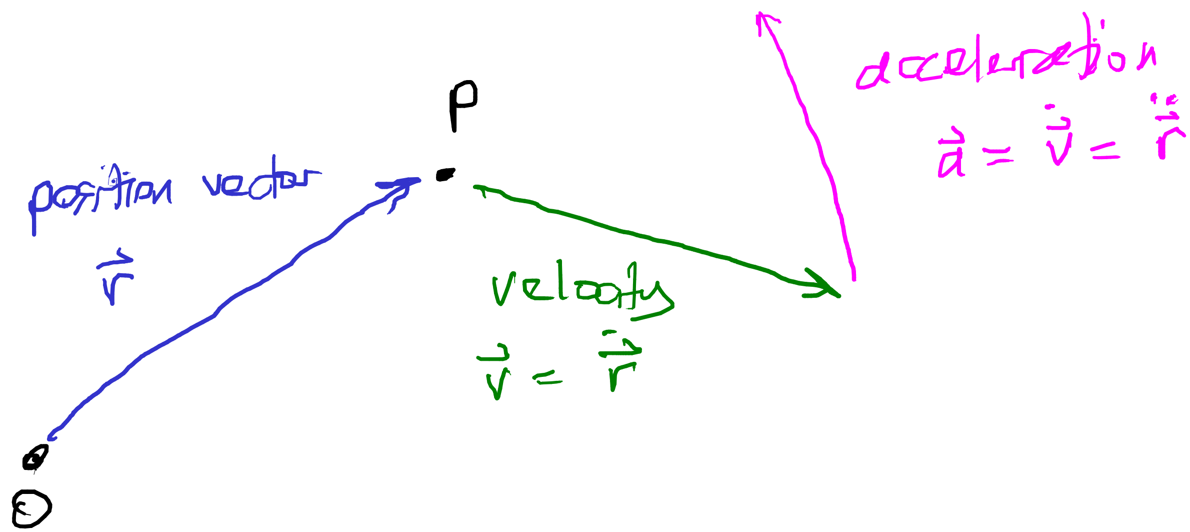
A. getting longer and rotating clockwise

B. getting shorter and rotating clockwise

C. getting longer and rotating counterclockwise

D. getting shorter and rotating counterclockwise

Today: - position, velocity, acceleration
- understand length and direction changes
- differentiate in polar coordinates
- chain rule for motion on paths



Changing length and direction

1

$$\vec{a} = a\hat{a}$$

$$\dot{\vec{a}} = \underbrace{\dot{a}\hat{a}}_{2} + \underbrace{a\dot{\hat{a}}}_{3}$$

2

$$a = \sqrt{\vec{a}\cdot\vec{a}} = (\vec{a}\cdot\vec{a})^{1/2}$$

$$\dot{a} = \frac{1}{2}(\vec{a}\cdot\vec{a})^{-1/2}\, 2\dot{\vec{a}}\cdot\vec{a}$$

$$= \frac{\dot{\vec{a}}\cdot\vec{a}}{a}$$

$$\boxed{\dot{a} = \dot{\vec{a}}\cdot\hat{a}}$$

$$\boxed{\begin{gathered}\dot{\vec{a}}\cdot\vec{a} + \vec{a}\cdot\dot{\vec{a}} \\ \dot{\vec{a}}\times\vec{a} + \vec{a}\times\dot{\vec{a}}\end{gathered}}$$

$$(\vec{a}\cdot\vec{a})^{-1/2} = \frac{1}{(\vec{a}\cdot\vec{a})^{1/2}}$$

$$= \frac{1}{\sqrt{\vec{a}\cdot\vec{a}}}$$

$$= \frac{1}{a}$$

3

$$\hat{a}\cdot\hat{a} = 1$$

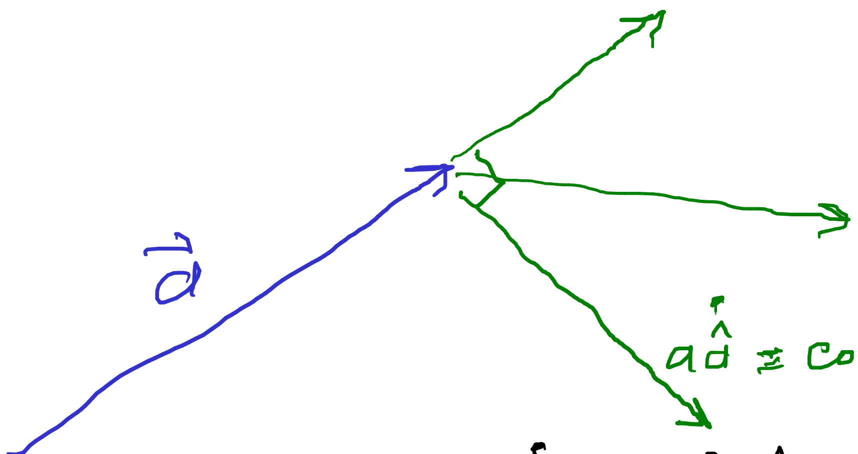
$$2\dot{\hat{a}}\cdot\hat{a} = 0$$

$$\boxed{\dot{\hat{a}}\cdot\hat{a} = 0}$$

unit vector derivatives are always orthogonal

$$\dot{a}\hat{a} = (\dot{\vec{a}}\cdot\hat{a})\hat{a} = \mathrm{proj}(\dot{\vec{a}},\vec{a})$$

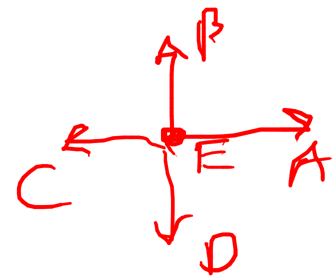
on $\hat{a}$



$\vec{a}$

$\dot{\vec{a}}$

$$a\dot{\hat{a}} = \mathrm{comp}(\dot{\vec{a}},\vec{a})$$



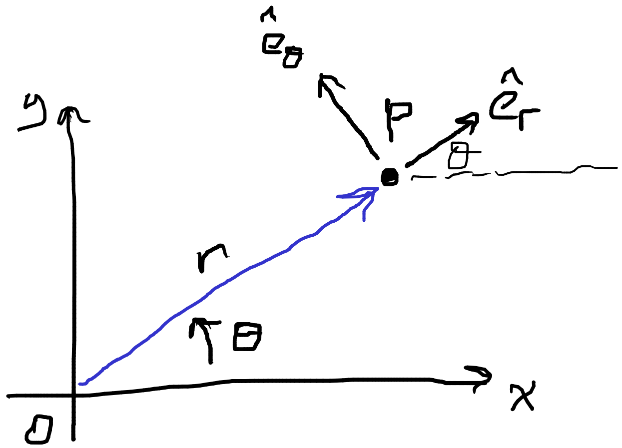
$$\dot{\vec{a}} = \dot{a}\hat{a} + a\dot{\hat{a}}$$

in dir $\hat{a}$ — orthogonal to $\hat{a}$

$\mathrm{proj}(\dot{\vec{a}},\vec{a})$ — $\mathrm{comp}(\dot{\vec{a}},\vec{a})$

change in length — change in dir

Derivatives in polar coordinates



$$\vec{r} = r\hat{e}_r$$

$$\hat{e}_r = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}$$

$$\hat{e}_\theta = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}$$

$$\vec{v} = \dot{\vec{r}} = \dot{r}\,\hat{e}_r + r\,\underline{\dot{\hat{e}}_r}$$

$\dot{\hat{e}}_r =$

A. $r\dot{\theta}\hat{e}_\theta$

B. $-r\dot{\theta}\hat{e}_\theta$

C. $\dot{\theta}\hat{e}_\theta$

D. $-\dot{\theta}\hat{e}_\theta$

$$\dot{\hat{e}}_r = \frac{d}{dt}\hat{e}_r$$

$$= \frac{d}{dt}\left(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}\right)$$

$$= \frac{d}{dt}(\cos\theta)\,\hat{\imath} + \frac{d}{dt}(\sin\theta)\,\hat{\jmath}$$

$$= \frac{d}{d\theta}(\cos\theta)\frac{d\theta}{dt}\,\hat{\imath} + \frac{d}{d\theta}(\sin\theta)\frac{d\theta}{dt}\,\hat{\jmath}$$

$$= -\sin\theta\,\dot{\theta}\,\hat{i} + \cos\theta\,\dot{\theta}\,\hat{j}$$

$$= \dot{\theta}\left(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j}\right)$$

$$= \dot{\theta}\,\hat{e}_\theta$$

$$\frac{d}{dt}(\cos\theta) = \frac{d}{dt}\left(\cos\theta(t)\right)$$

$$= -\sin\theta(t)\,.\,\dot{\theta}(t)$$

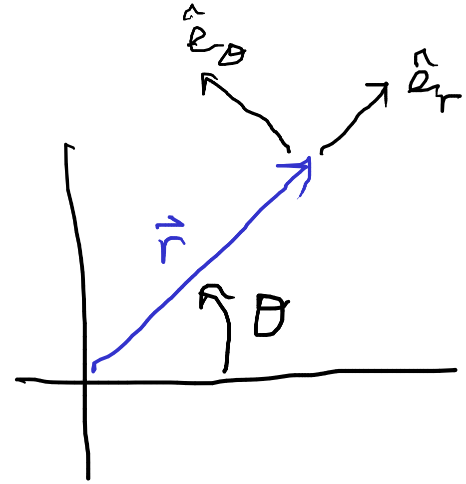
$\dot{\hat{e}}_\theta =$

A. $r\dot{\theta}\,\hat{e}_r$

B. $-r\dot{\theta}\,\hat{e}_r$

C. $\dot{\theta}\,\hat{e}_r$

D. $-\dot{\theta}\,\hat{e}_r$



Notation: $\dot{\theta} = \omega$ angular velocity

rad/s