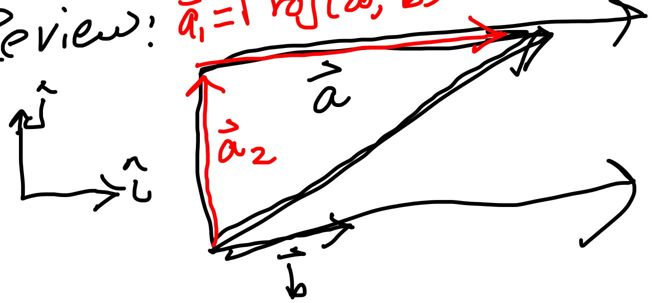
Class 4: ① angular velocity
② derivatives of vectors

Review: $\vec{a}_1 = \text{Proj}(\vec{a}, \vec{b})$



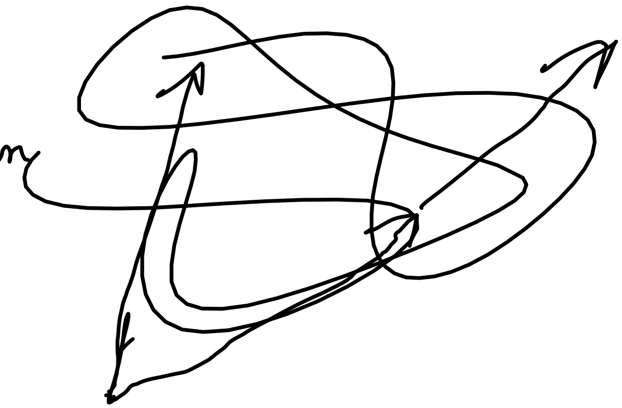
$\text{Proj}(\vec{a}, \vec{b})$

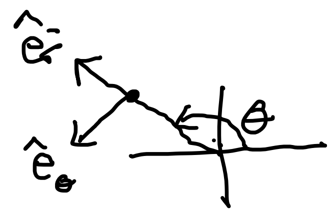
$\vec{a}_1 + \vec{a}_2 = \vec{a}$

$\swarrow \text{Proj}(\vec{a}, \vec{b}) \qquad \searrow \text{Comp}(\vec{a}, \vec{b})$

$$\text{Proj}(\vec{a}, \vec{b}) = \underbrace{(\vec{a} \cdot \hat{b})}_{\text{scalar}} \underbrace{\hat{b}}_{\text{direction}}$$

$$\hat{b} = \vec{b} / b$$





② 2D coord. systems

	cartesian	polar
coordinates	x, y	r, θ
basis vectors	$\hat{i}, \hat{j}$	$\hat{e}_r, \hat{e}_\theta$
position	$\vec{r} = x\hat{i} + y\hat{j}$	$\vec{r} = r\hat{e}_r$
today velocity		

$$\hat{e}_r = \cos\theta\, \hat{i} - \sin\theta\, \hat{j}$$
$$\hat{e}_\theta = \sin\theta\, \hat{i} + \cos\theta\, \hat{j}$$

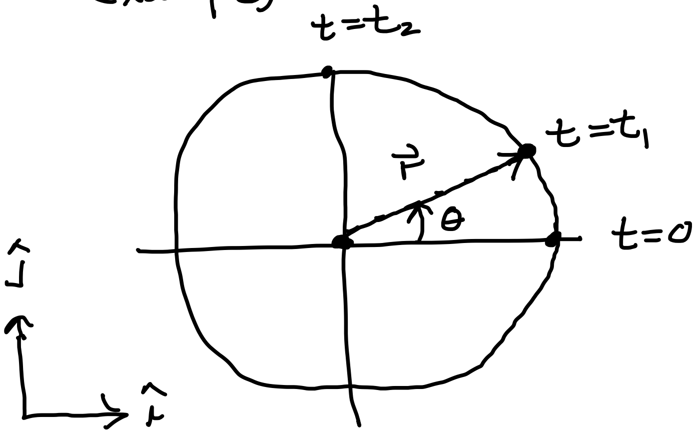
Angular Velocity

A rotation is defined by

1. how fast $\longrightarrow$ magnitude
2. about which axis $\longrightarrow$ direction

} rotation is a vector.

example)



$\vec{r}(t)$, a vector pinned at origin, rotates in the plane, path of the tip traces out a circle.

As $\vec{r}$ rotates: in polar coords

1. radius r = constant
2. $\theta(t)$ = depends on time

DEFINE $\boxed{\omega = \frac{d\theta}{dt} = \dot{\theta}}$ rate of rotation

axis of rotation: $\hat{k}$

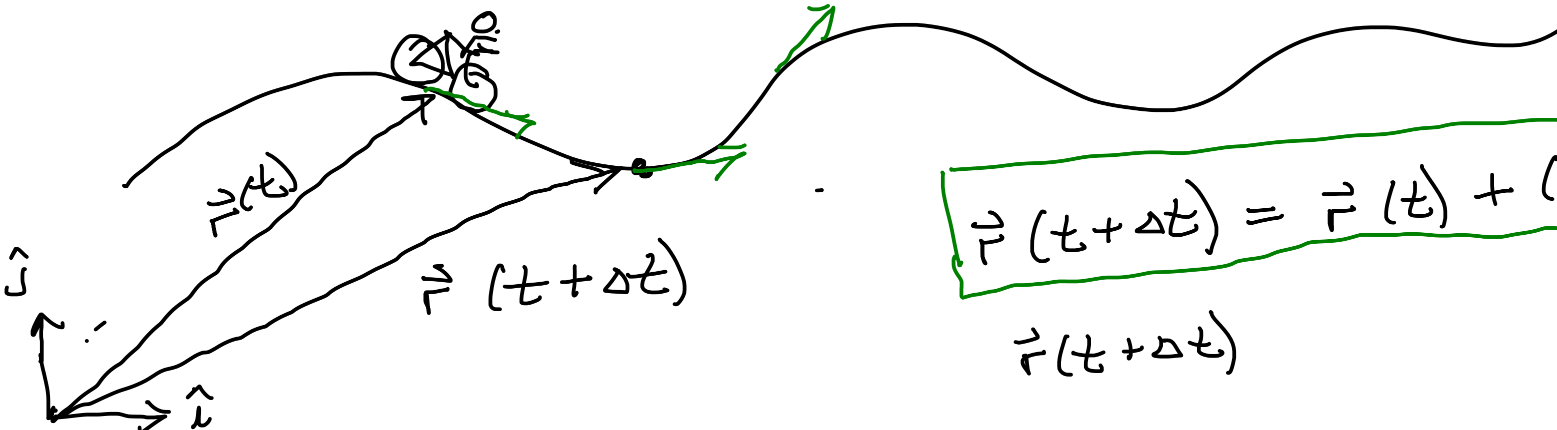
vector angular velocity $\vec{\omega} = \omega \hat{k}$

$= \dot{\theta}\hat{k}$

convention:

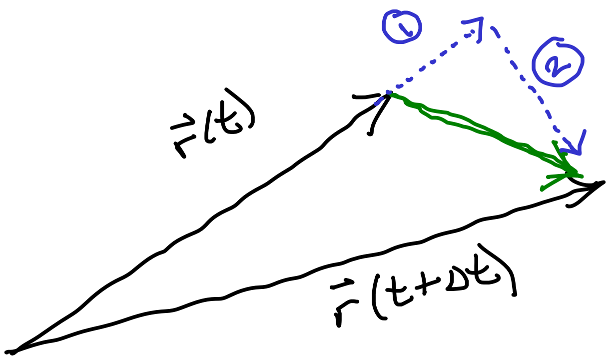
CCW	$\omega > 0$
CW	$\omega < 0$

Time Derivative of a Vector



$$\boxed{\vec{r}(t+\Delta t) = \vec{r}(t) + (\Delta t)\frac{d\vec{r}}{dt}}$$

$\vec{r}(t+\Delta t)$



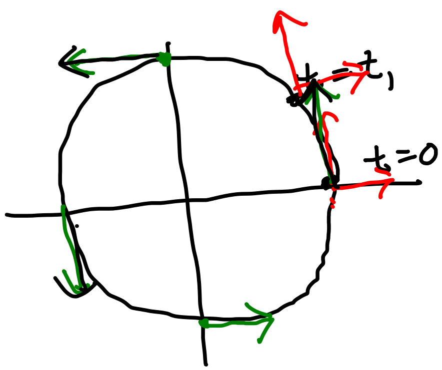
$\frac{d\vec{r}}{dt}$ is a vector

1. change in magnitude of $\vec{r}(t)$
2. change in direction of $\vec{r}(t)$

$\vec{a} = a\hat{a}$

$$\frac{d\vec{a}}{dt} = \dot{\vec{a}} = \underbrace{\left(\frac{da}{dt}\right)\hat{a}}_{\substack{\text{parallel to } \vec{a}\text{, change in}\\ \text{magn. of } \vec{a}}} + \underbrace{a\left(\frac{d\hat{a}}{dt}\right)}_{\substack{\text{perpendicular to } \vec{a}\text{,}\\ \text{changes in direction } \vec{a}}}$$

PRODUCT RULE



	CARTESIAN	POLAR
position	$\vec{r} = x\hat{i} + y\hat{j}$	$\vec{r} = r\hat{e}_r$
velocity	?	?

in cartesian:

$$\vec{v} = \dot{\vec{r}} = \frac{d}{dt}\left(x\hat{i} + y\hat{j}\right)$$

$$= \frac{dx}{dt}\hat{i} + \cancel{x\frac{d\hat{i}}{dt}} + \frac{dy}{dt}\hat{j} + \cancel{y\frac{d\hat{j}}{dt}}$$

$$= \dot{x}\hat{i} + \dot{y}\hat{j}$$

$$= -r\omega\sin(\omega t)\hat{i} + r\omega\cos(\omega t)\hat{j}$$

$x = r\cos\theta = r\cos\omega t$

$y = r\sin\theta = r\sin\omega t$

in polar coordinates:

$$\vec{r} = r\hat{e}_r$$

$$\vec{v} = \dot{\vec{r}} = \frac{dr}{dt}\hat{e}_r + r\dot{\hat{e}}_r$$

$$= \underbrace{\dot{r}\hat{e}_r}_{\text{parallel to } \vec{r}} + \underbrace{r\dot{\hat{e}}_r}_{\to \text{ perpendicular to } \hat{e}_r \text{ (we will see)}}$$

$$= r\dot{\hat{e}}_r \qquad \text{since } \dot{r} = 0$$

$$\hat{e}_r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$

$$= \cos(\omega t)\,\hat{i} + \sin(\omega t)\,\hat{j}$$

$$\dot{\hat{e}}_r = -\omega\sin\omega t\,\hat{i} + \omega\cos(\omega t)\,\hat{j}$$

$$= \omega(-\sin\omega t\,\hat{i} + \cos\omega t\,\hat{j})$$

$$= \omega\hat{e}_\theta$$

$$\vec{v} = r\omega\hat{e}_\theta$$