TAM 212 – Dynamics

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Summer 2019

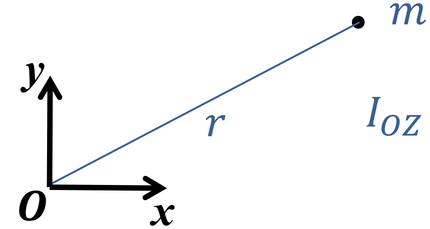
Recap

- Center of Mass, Moments of Inertia

Today

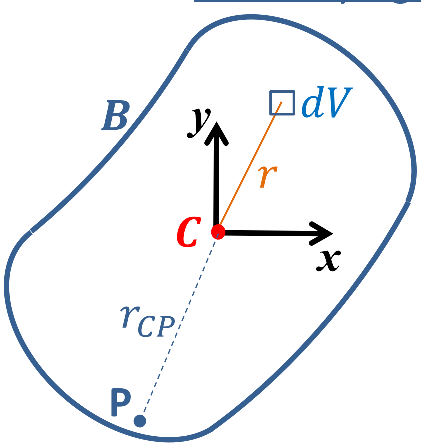
- Moments of Inertia, Kinetics of Rigid Bodies

Point mass



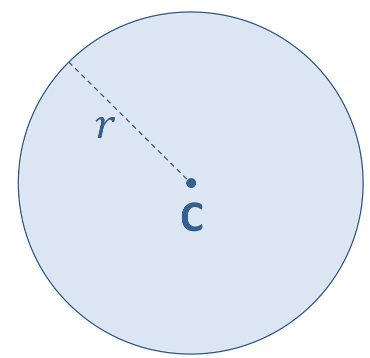
$$I_{OZ} = mr^2$$

Arbitrary rigid body



$$I_{CZ} = \iiint_B \rho r^2 dV$$

Simple shapes: tables



Disk:

$$I_{CZ} = \frac{1}{2} mr^2$$

r_{CP} = distance orthogonal to Z-axis

Parallel Axis Theorem $\quad I_{PZ} = I_{CZ} + mr_{CP}^2 \quad$ must be from COM

Additive Theorem $\quad I_{PZ}^{total} = I_{PZ}^{B_1} + I_{PZ}^{B_2} \quad$ must be on the same point, along the same axis

First use Parallel Axis Theorem, then Additive Theorem

Rigid Body Kinetics

Kinematics $\longrightarrow$ $\vec{r}, \vec{v}, \vec{a}, \vec{\omega}, \vec{\alpha}$ (no forces)

Kinetics $\longrightarrow$ $\vec{F} = m\vec{a}, \vec{M} = I\vec{\alpha}$, forces, moments

Particle: $\Sigma\vec{F} = m\vec{a}$

Equations of motion

Rigid Body:

$$\boxed{\begin{aligned} \Sigma\vec{F} &= m\vec{a}_C \\ \Sigma M_{CZ} &= I_{CZ}\alpha_Z \end{aligned}}$$

C = CoM = Center of Mass

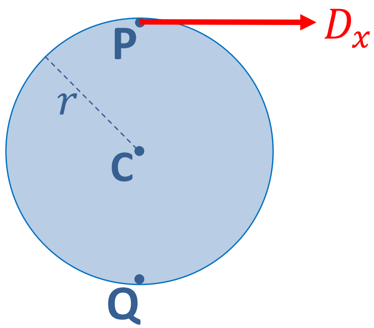
o

$$\boxed{\Sigma M_{OZ} = I_{OZ}\alpha_Z}$$

O = Fixed Point

$$\boxed{\text{Forces \& Moments}} \xleftrightarrow[m, I]{\text{inertia}} \boxed{\begin{matrix}\text{Motion} \\ \vec{a}, \vec{\alpha}\end{matrix}}$$

Example



Disk starts at rest.
No gravity.
Force $D_x > 0$ applied at P

Initially

P accelerates

A. left
B. zero
C. right

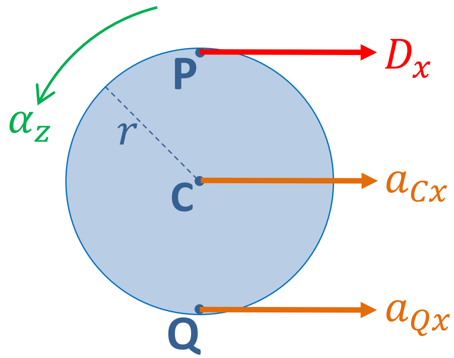
C accelerates

A. left
B. zero
C. right

Q accelerates

A. left
B. zero
C. right

Example



Disk starts at rest.
No gravity.
Force $D_x > 0$ applied at P

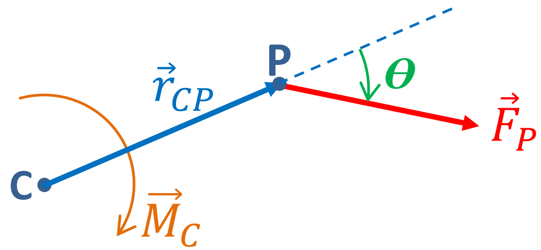
$a_{Cx} =$
A. $\frac{2D_x}{m}$
B. $\frac{D_x}{m}$
C. 0
D. $\frac{-D_x}{m}$
E. $\frac{-2D_x}{m}$

$\alpha_z =$
A. $\frac{2D_x}{mr}$
B. $\frac{D_x}{mr}$
C. 0
D. $\frac{-D_x}{mr}$
E. $\frac{-2D_x}{mr}$

$a_{Qx} =$
A. $\frac{2D_x}{m}$
B. $\frac{D_x}{m}$
C. 0
D. $\frac{-D_x}{m}$
E. $\frac{-2D_x}{m}$

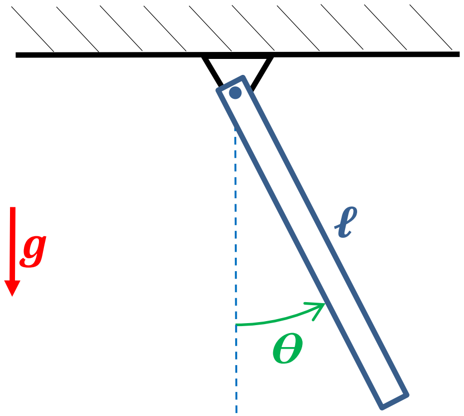
Moments

$\boxed{\overrightarrow{M}_C = \vec{r}_{CP} \times \vec{F}_P}$ moment about C due to force at P



Example: Rigid-rod Pendulum

1. System diagram



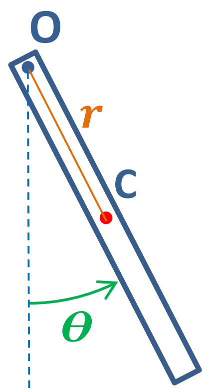
Rigid rod, length ℓ, mass m, gravity g

Find: angular acceleration as a function of θ, $\dot{\theta}$

2. FBDs

Example: Rigid-rod Pendulum

3. Kinematics: $\vec{a}_C, \vec{\alpha}$



Example: Rigid-rod Pendulum

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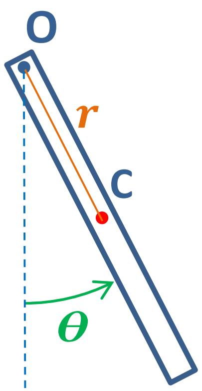


$\alpha_Z =$ **A.** $r\ddot{\theta}$

B. $-r\dot{\theta}^2$

C. $\ddot{r}$

D. $2r\dot{\theta}^2$

E. $\ddot{\theta}$

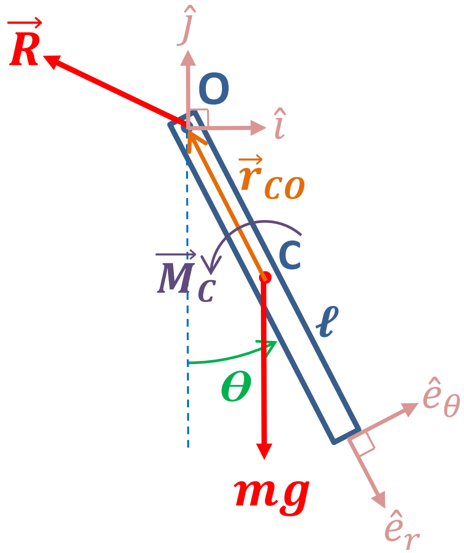
Coordinates for $\vec{a}_C$?

A. Cartesian

B. Polar

Example: Rigid-rod Pendulum

4. Kinetics (Newton):



$$\vec{R} = R_x\hat{\imath} + R_y\hat{\jmath}$$

$$= R_r\hat{e}_r + R_\theta\hat{e}_\theta$$

$M_{CZ} =$

A. $\frac{\ell}{2}R_r$

B. $\frac{\ell}{2}R_\theta$

C. $-\frac{\ell}{2}R_r$

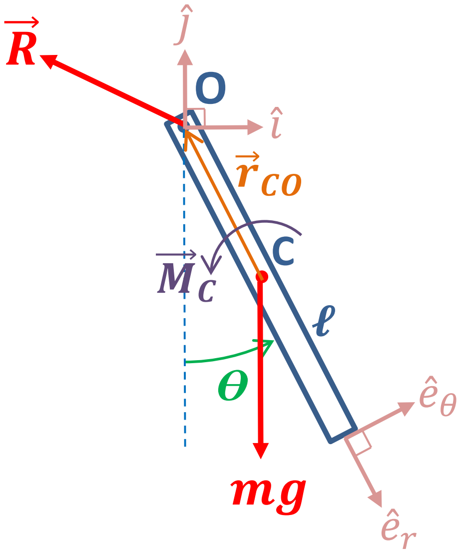
D. $-\frac{\ell}{2}R_\theta$

E. NOTA

Example: Rigid-rod Pendulum

4. Kinetics (Newton):

$$\vec{R} = R_r \hat{e}_r + R_\theta \hat{e}_\theta \qquad \vec{a} = (\overset{0}{\cancel{\ddot{r}}} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\overset{0}{\cancel{\dot{r}}}\dot{\theta})\hat{e}_\theta$$



$$\Sigma\vec{F} = m\vec{a}_C \quad \Rightarrow \vec{R} - mg\hat{\jmath} = -m\frac{\ell}{2}\dot{\theta}^2\hat{e}_r + m\frac{\ell}{2}\ddot{\theta}\hat{e}_\theta \qquad (1)$$

$$\Sigma M_{CZ} = I_{CZ}\alpha_Z \qquad I_{CZ} = \frac{1}{12}m\ell^2$$

$$\vec{M}_C = \vec{r}_{CO} \times \vec{R} = \left(-\frac{\ell}{2}\hat{e}_r\right) \times (R_r\hat{e}_r + R_\theta\hat{e}_\theta) = -\frac{\ell}{2}R_\theta\hat{k}$$

$$M_{CZ} = I_{CZ}\alpha_Z \Rightarrow -\frac{\ell}{2}R_\theta = \frac{1}{12}m\ell^2\ddot{\theta} \qquad (2)$$

5. Algebra

(1) θ-direction $\Rightarrow$ $R_\theta - mg\sin\theta = m\frac{\ell}{2}\ddot{\theta}$ (3)

(2) $\Rightarrow$ $-\ell R_\theta = \frac{1}{6}m\ell^2\ddot{\theta}$ (4)

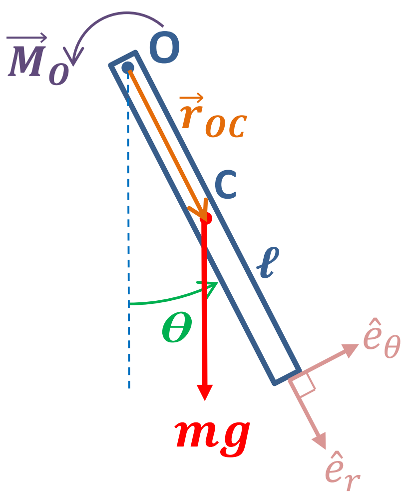
ℓ(3)+(4) $\Rightarrow$ $-mg\ell\sin\theta = \frac{2}{3}m\ell^2\ddot{\theta}$

$$\boxed{\ddot{\theta} = -\frac{3}{2}\frac{g}{\ell}\sin\theta}$$

Solve this numerically

Example: Rigid-rod Pendulum
Alternative solution using fixed-point O

4. Kinetics (Newton): $\vec{a} = (\cancelto{0}{\ddot{r}} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\cancelto{0}{\dot{r}}\dot{\theta})\hat{e}_\theta$



$$\Sigma M_{OZ} = I_{OZ}\alpha_Z \qquad I_{OZ} = \frac{1}{3}m\ell^2$$

$$\vec{M}_O = \vec{r}_{OC} \times \vec{F}_g = \left(\frac{\ell}{2}\hat{e}_r\right) \times (mg\cos\theta\,\hat{e}_r - mg\sin\theta\,\hat{e}_\theta)$$

$$\vec{M}_O = -mg\frac{\ell}{2}\sin\theta\,\hat{k}$$

$$M_{OZ} = I_{OZ}\alpha_Z \Rightarrow -mg\frac{\ell}{2}\sin\theta = \frac{1}{3}m\ell^2\ddot{\theta}$$

5. Algebra

$$\boxed{\ddot{\theta} = -\frac{3}{2}\frac{g}{\ell}\sin\theta}$$

Instantaneous Centers (IC)

Rigid body moving in 2D.

- Rotating and possible translating

IC = M = point that is not moving
= point that RB is rotating about
(only defined instantaneously)

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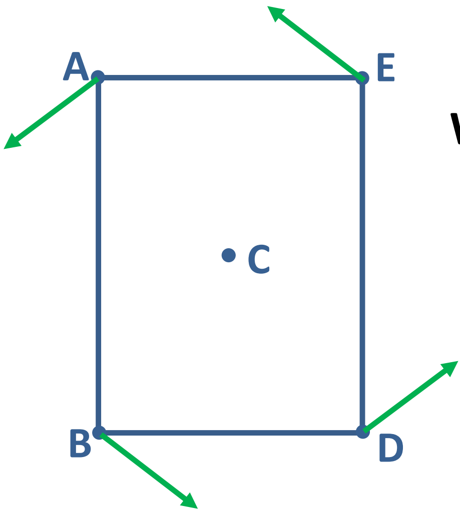

Which point is M?

Instantaneous Centers (IC)

Rigid body moving in 2D.

- Rotating and possible translating

IC = M = point that is not moving
= point that RB is rotating about
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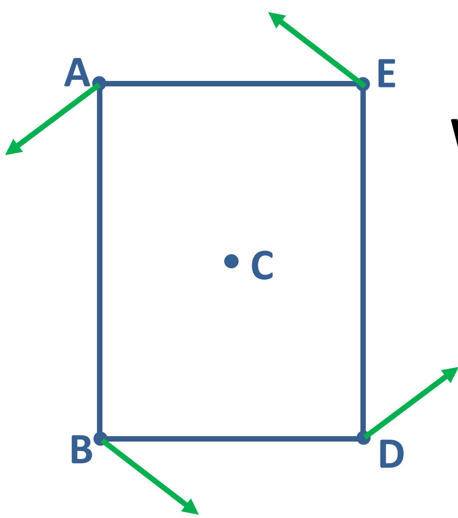
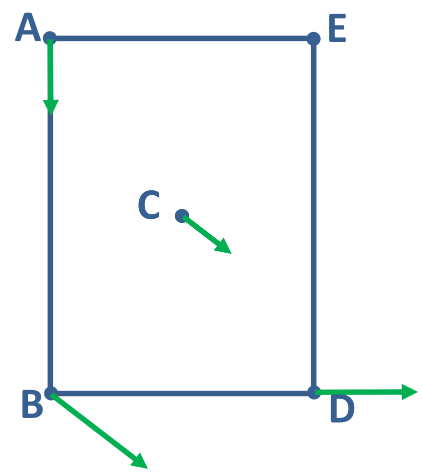
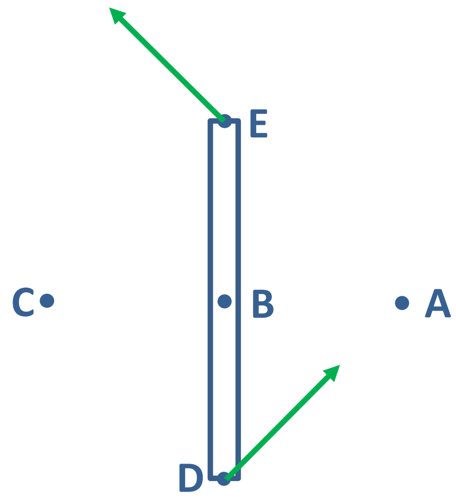


Which point is M?

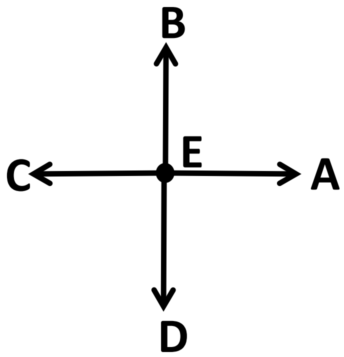


IC Examples

Which point is M?

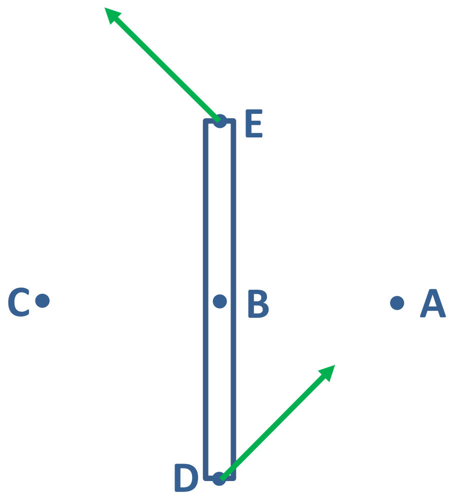


Direction of $\vec{v}_B$?



IC Examples

Which point is M?



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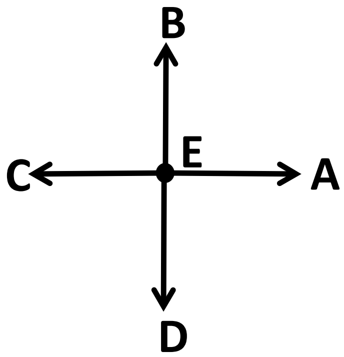
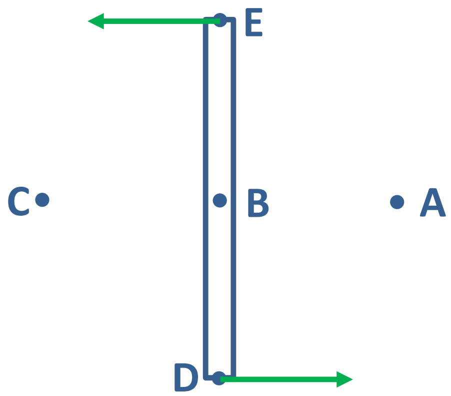


Which point is M?



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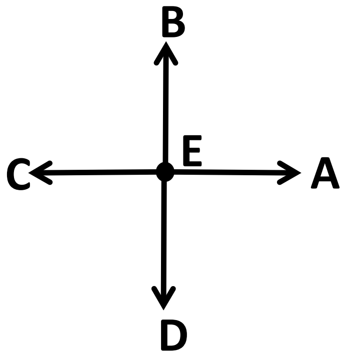
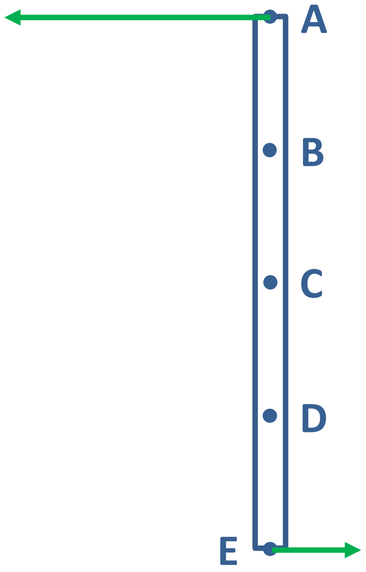


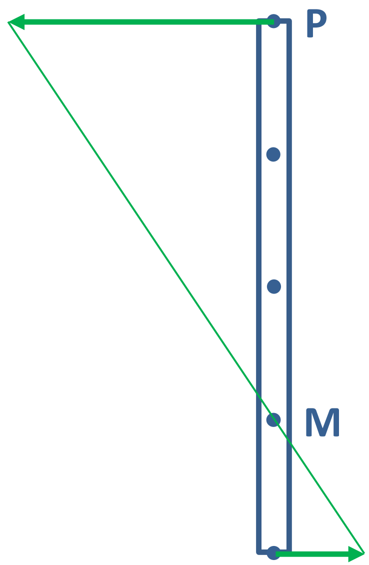
IC Examples

Which point is M?



Intuition for IC

From M, $\vec{v}_P$ is orthogonal to $\vec{r}_{MP}$ (why?)

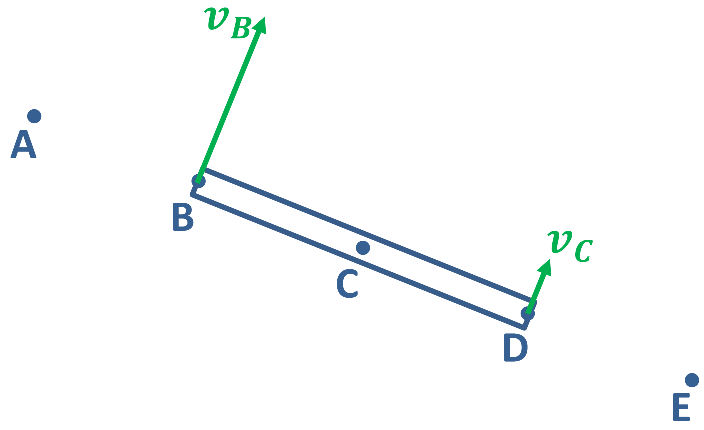


$v_P = \omega r_{MP}$

speed proportional to distance from M

More IC Examples

Which point is M?



More IC Examples

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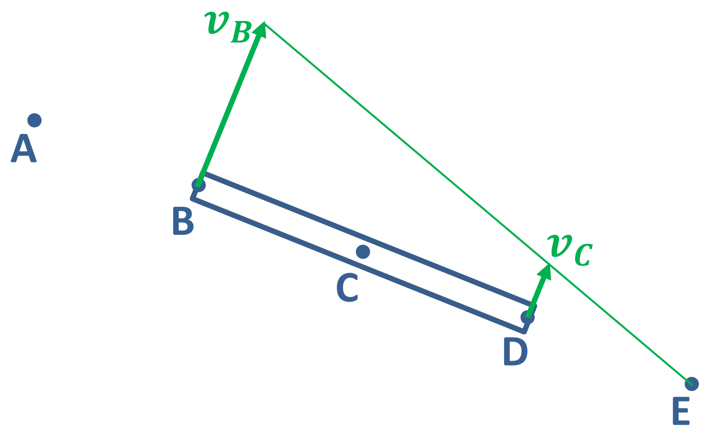
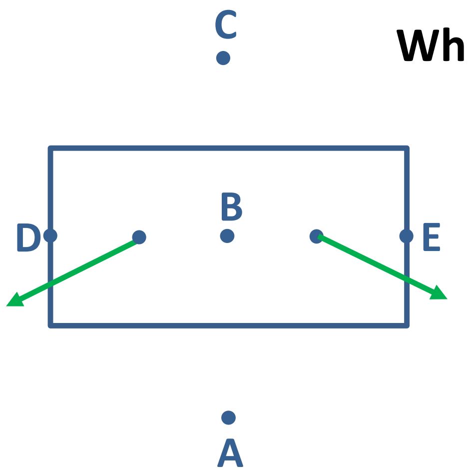
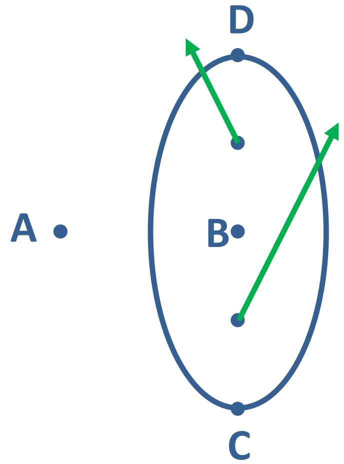


Which point is M?



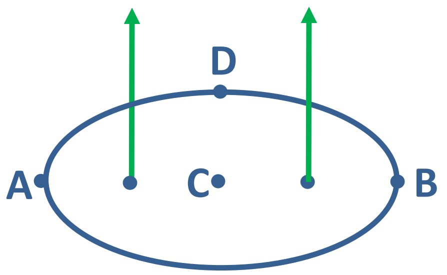
Even More IC Examples

Which point is M?



E - none

Which point is M?



E - none

Graphical Rules for Finding M (the IC)

<u>Draw lines perpendicular to velocities</u>

- If the lines intersect at a single point

  $\Rightarrow$ that point is M
- If the lines are the same lines

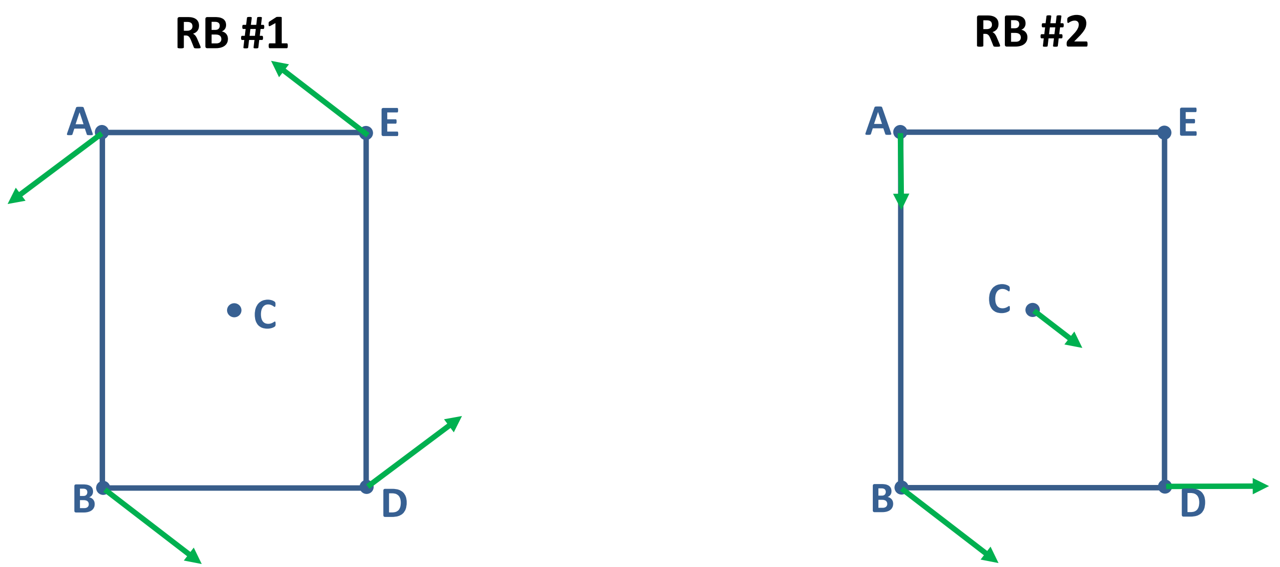
  $\Rightarrow$ Draw a line through the velocity tips
  - If the lines intersect at a single point

    $\Rightarrow$ that point is M

<u>Careful!</u>

- Consistent direction of rotation
- Consistent speeds $v = \omega r$
- Body may not be rotating (pure translation)

Example



Which body is translating?

A. RB #1
B. RB #2
C. both
D. neither
E. can't tell

Some Key Points

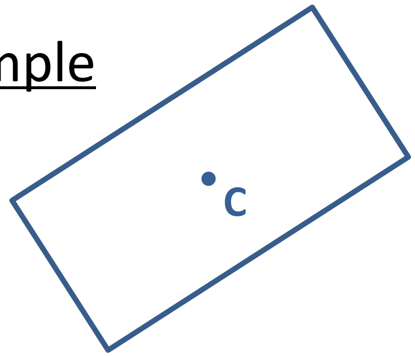
- Rotation is a property of the body as a whole
- Translation really isn't such a property
  (unless we have pure translation)
- We can talk about translation of specific points, like center C

Calculating the Position of M

Key property: M is the point with $\vec{v}_M = 0$ at that instant

OR $\vec{v}_P$ is in pure rotation about M.

Example

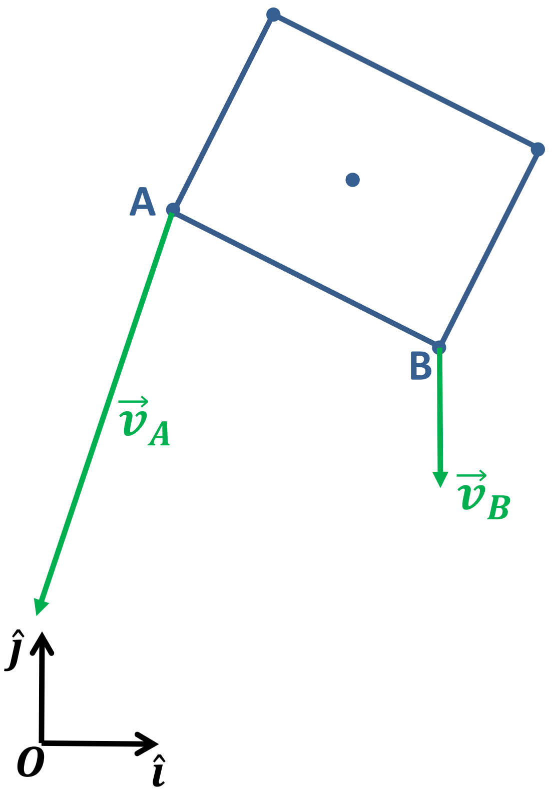


$\vec{v}_C = (4, 2)\mathrm{m/s}$

$\vec{\omega} = -2\ \mathrm{rad/s}$

Where is M relative to C ?

Example

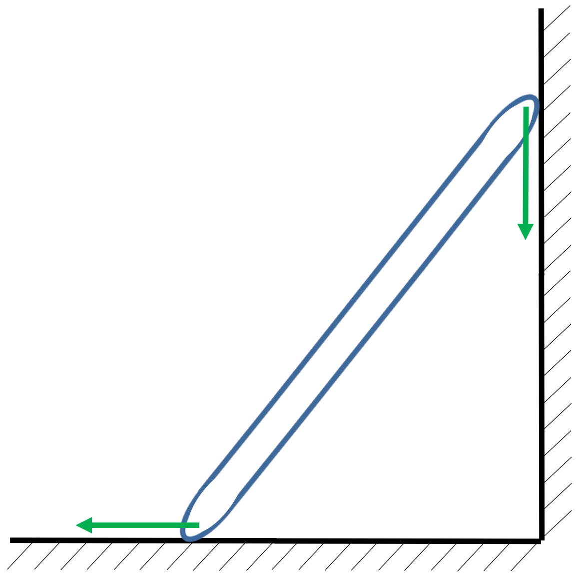


$\vec{r}_A = (1, 4)\mathrm{m}$ $\quad$ $\vec{v}_A = (-3, -9)\mathrm{m/s}$

$\vec{r}_B = (3, 3)\mathrm{m}$ $\quad$ $\vec{v}_B = (0, -3)\mathrm{m/s}$

Where is M?

Example



The Ladder is sliding down the wall.