TAM 212 – Dynamics

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Particle Kinetics (Recap)

Statics

$\Sigma\vec{F} = 0$

Kinematics

$\vec{a}$

Kinetics

$\Sigma\vec{F} = m\vec{a}$

How much force is causing motion?	How much mass is accelerating?	What is the acceleration?

Method of assumed forces (simulation): know $\vec{F} \rightarrow$ find $\vec{a}$

Method of assumed motion (measurement): know $\vec{a} \rightarrow$ find $\vec{F}$

Rigid Body Kinetics

Kinematics $\longrightarrow$ $\vec{r}, \vec{v}, \vec{a}, \vec{\omega}, \vec{\alpha}$ (no forces)

Kinetics $\longrightarrow$ $\vec{F} = m\vec{a}, \vec{M} = I\vec{\alpha}$, forces, moments

Particle: $\Sigma\vec{F} = m\vec{a}$

Rigid Body:

$$\boxed{\begin{aligned} \Sigma\vec{F} &= m\vec{a}_C \\ \Sigma M_{CZ} &= I_{CZ}\alpha_Z \end{aligned}}$$

C = CoM = Center of Mass

o

$$\boxed{\Sigma M_{OZ} = I_{OZ}\alpha_Z}$$

O = Fixed Point

$$\boxed{\text{Forces \& Moments}} \xleftrightarrow[m, I]{\text{inertia}} \boxed{\begin{matrix}\text{Motion} \\ \vec{a}, \vec{\alpha}\end{matrix}}$$

Solution steps

Method of assumed forces: $\vec{F}, \vec{M} \rightarrow \vec{a}, \vec{\alpha}$ (simulation)

Method of assumed motion: $\vec{a}, \vec{\alpha} \rightarrow \vec{F}, \vec{M}$ (measurement)

<u>Solution steps</u>

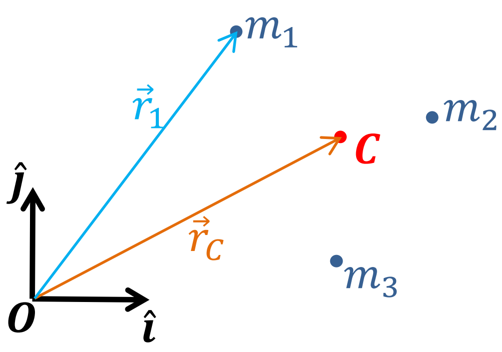
1. System diagram – coordinates
2. Free body diagrams (FBD) – forces and moments
3. Kinematics $\rightarrow \vec{a}, \vec{\alpha}$
4. Kinetics: Newton $\vec{F} = m\vec{a},\ M_{CZ} = I_{CZ}\alpha_Z$
5. Algebra: solve for $\vec{F}, \vec{M}$ or $\vec{a}, \vec{\alpha}$

Mass

m_i = resistance to acceleration (inertial mass) $\quad F = m_i a$

m_g = source of gravity (gravitational mass) $\quad F = G\dfrac{Mm_g}{r^2}$

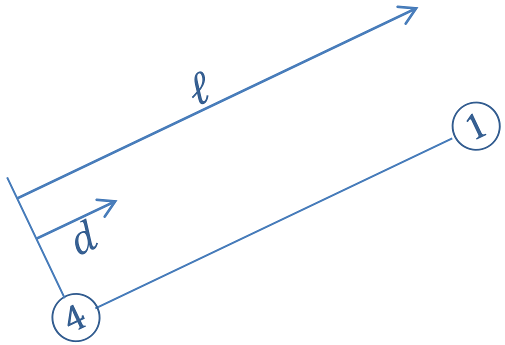
CoM (Center of Mass)



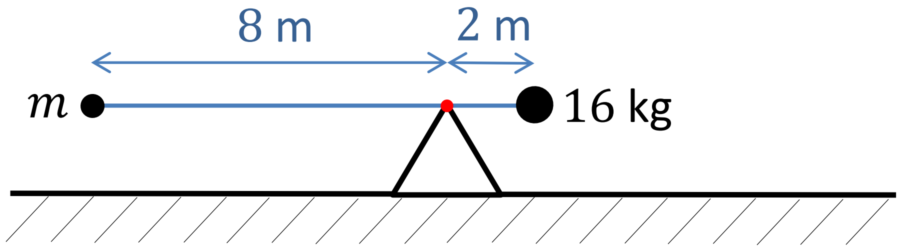
$m = \sum_i m_i$ total mass

$\vec{r}_C = \frac{1}{m}\sum_i m_i \vec{r}_i$

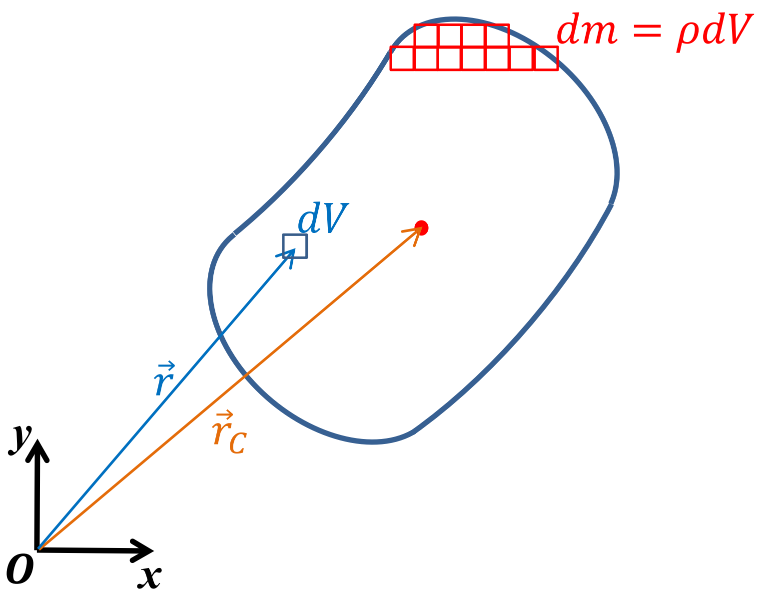
Example



Example



CoM for Rigid Bodies



$$\boxed{\begin{aligned} m &= \iiint_B \rho dV \\ \vec{r}_C &= \frac{1}{m} \iiint_B \rho \vec{r} dV \end{aligned}}$$

$$dV = dxdydz$$

$$\vec{r} = (x, y, z)$$
$$= x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

Units of density?

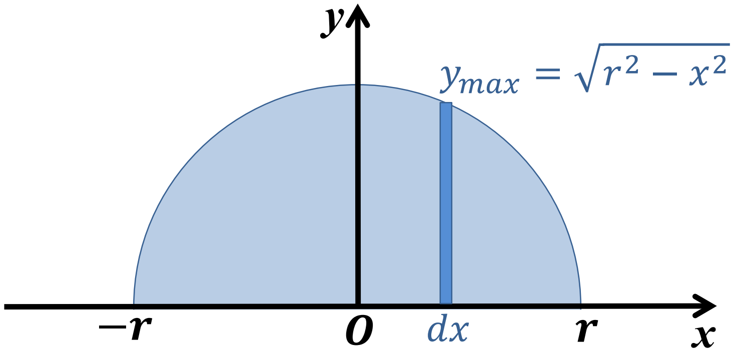
$[\rho] = kg^{\square} m^{\square}$

$kg^1 m^{-3}$ → volumetric density

$kg^1 m^{-2}$ → areal density

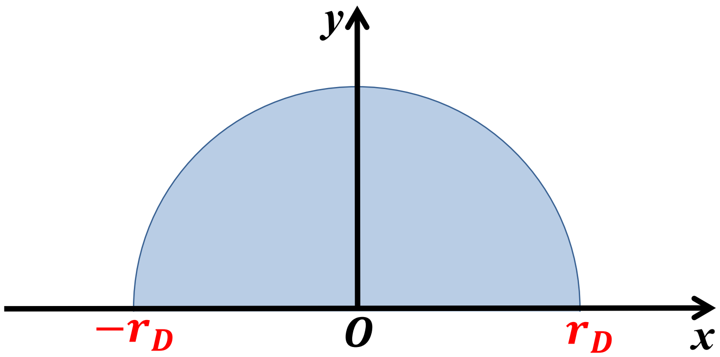
$kg^1 m^{-1}$ → linear density

Example



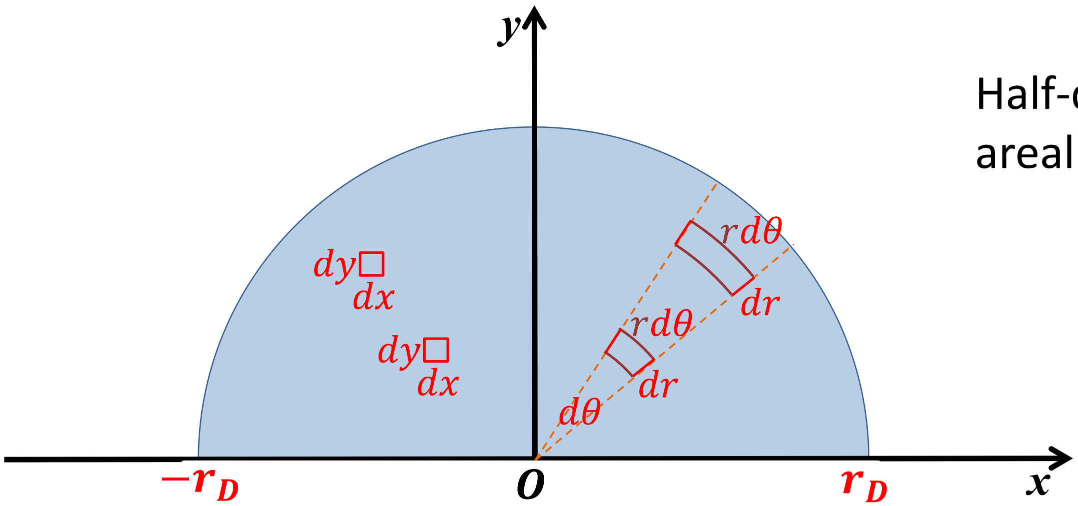
Half-disk, radius r , uniform areal density. What is $\vec{r}_C$?

Example – Polar Coordinates



Half-disk, radius r_D , uniform areal density. What is $\vec{r}_C$?

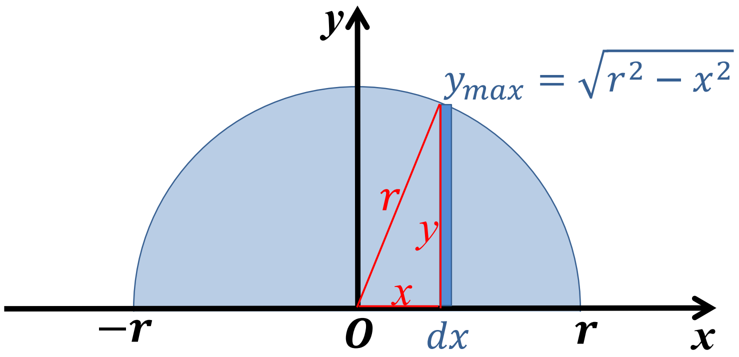
Example – Polar Coordinates



Half-disk, radius r_D , uniform areal density. What is $\vec{r}_C$?

$$m = \int_0^{r_D} \int_0^{\pi} \rho r d\theta dr$$

Example



Half-disk, radius r , uniform areal density. What is $\vec{r}_C$?

$$m = \iint_B \rho dA$$

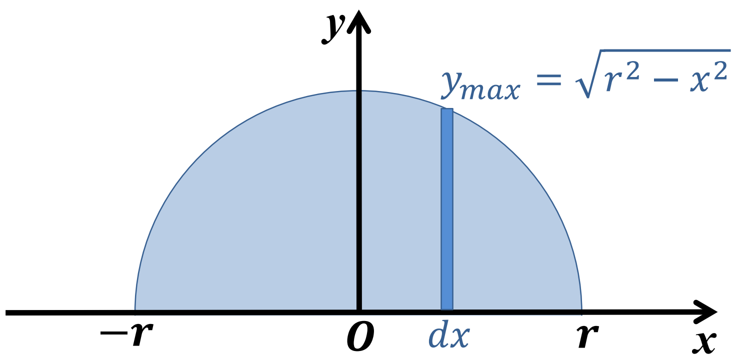
$$\vec{r}_C = \frac{1}{m}\iint_B \rho\vec{r}dA$$

$$m = \int_{-r}^{r}\int_{0}^{\sqrt{r^2-x^2}} \rho dydx$$

$$m = \rho\frac{1}{2}\pi r^2$$

$$\vec{r}_C = \frac{1}{m}\int_{-r}^{r}\int_{0}^{\sqrt{r^2-x^2}} \rho(x\hat{\imath} + y\hat{\jmath})dydx$$

Example



$$m = \frac{1}{2}\rho\pi r^2 \qquad r_{cx} = 0 \text{ (symmetry)}$$

$$\vec{r}_C = \frac{1}{m}\int_{-r}^{r}\int_{0}^{\sqrt{r^2-x^2}} \rho(x\hat{\imath} + y\hat{\jmath})dydx$$

$$= \frac{1}{m}\int_{-r}^{r}\left[\rho xy\hat{\imath} + \frac{1}{2}\rho y^2\hat{\jmath}\right]_0^{\sqrt{r^2-x^2}} dx$$

$$= \frac{1}{m}\int_{-r}^{r}\left(\rho x\sqrt{r^2 - x^2}\hat{\imath} + \frac{1}{2}\rho(r^2 - x^2)\hat{\jmath}\right)dx$$

$$= \frac{1}{m}\left[-\frac{\rho}{3}(r^2 - x^2)^{3/2}\hat{\imath} + \left(\frac{1}{2}\rho r^2 x - \frac{1}{6}\rho x^3\right)\hat{\jmath}\right]_{-r}^{r}$$

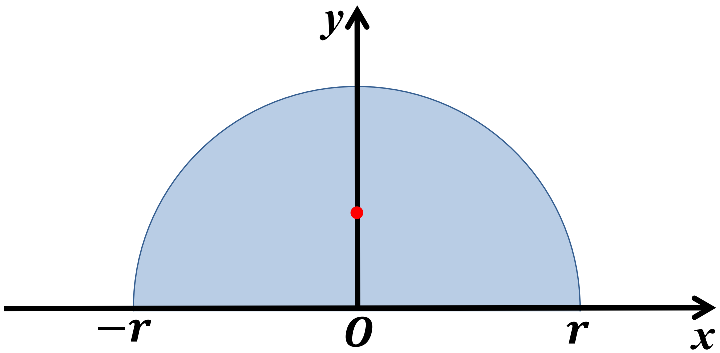
$$= \frac{1}{m}\left(\rho r^3 - \frac{1}{3}\rho r^3\right)\hat{\jmath}$$

$$= \frac{1}{m}\frac{2}{3}\rho r^3\hat{\jmath}$$

$$= \frac{1}{\frac{1}{2}\rho\pi r^2}\frac{2}{3}\rho r^3\hat{\jmath}$$

$$= \frac{4r}{3\pi}\hat{\jmath}$$

Example



Half-disk, radius r , uniform areal density. What is $\vec{r}_C$?

$$m = \frac{1}{2}\rho\pi r^2$$

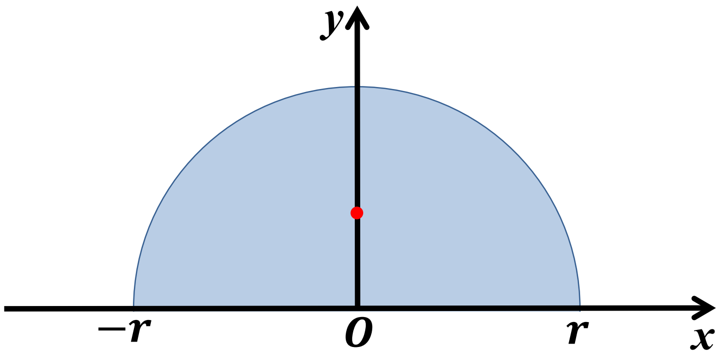
$$\vec{r}_C = \frac{1}{m}\int_{-r}^{r}\int_{0}^{\sqrt{r^2-x^2}}\rho[x,y]dydx$$

```
>> syms rho r x y
>> m = (1/2) * rho * pi * r^2;
>> rC = (1/m) * int(int(rho * [x,y], y, 0, sqrt(r^2 - x^2)), x, -r, r)

rC =

[ 0, (4*r)/(3*pi)]
```

Example



Half-disk, radius r , uniform areal density. What is $\vec{r}_C$?

$$m = \frac{1}{2}\rho\pi r^2$$

$$\vec{r}_C = \frac{1}{m}\int_{-r}^{r}\int_{0}^{\sqrt{r^2-x^2}} \rho[x,y]dy\, dx$$

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>> syms rho r x y
>> m = (1/2) * rho * pi * r^2;
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rC =

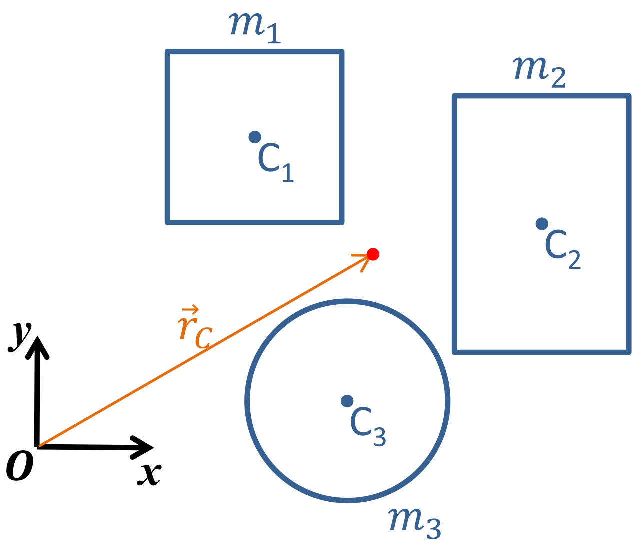
[ 0, (4*r)/(3*pi)]
```

Integrand var. LL UL

Integrand var. LL UL

The function f(x) to be integrated is called the integrand.

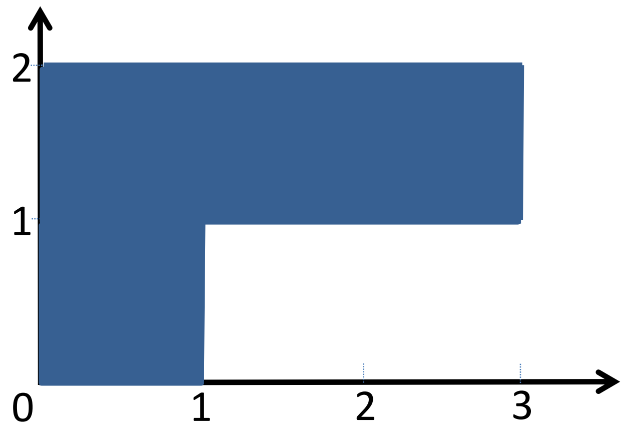
CoM for Composite Bodies



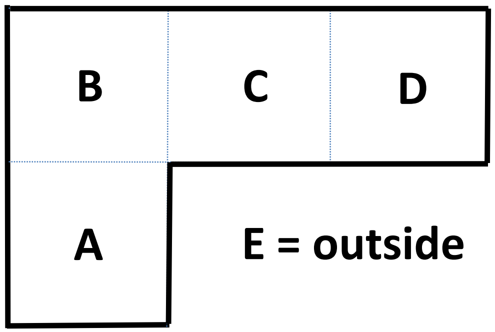
$$m = \Sigma m_i$$

$$\vec{r}_C = \frac{1}{m} \Sigma m_i \vec{r}_{C_i}$$

Example



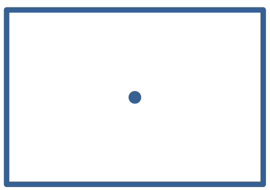
Constant density ρ .
What is $\vec{r}_C$?



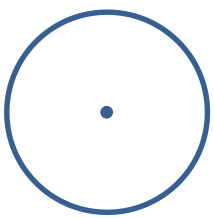
Which box contains the CoM?

Finding the CoM theoretically

<u>Simple shapes</u>



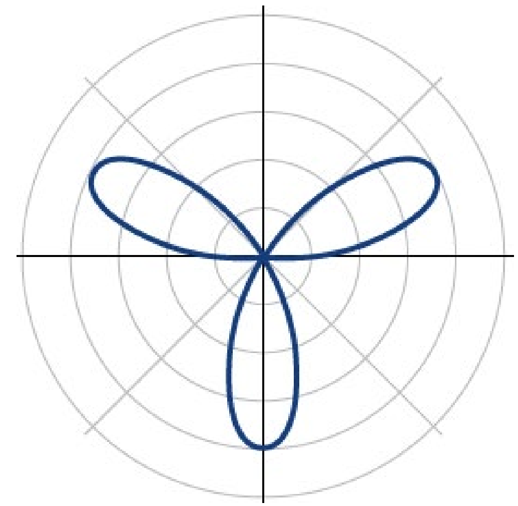
Use symmetry
or tables



Combination of <u>simple shapes</u>

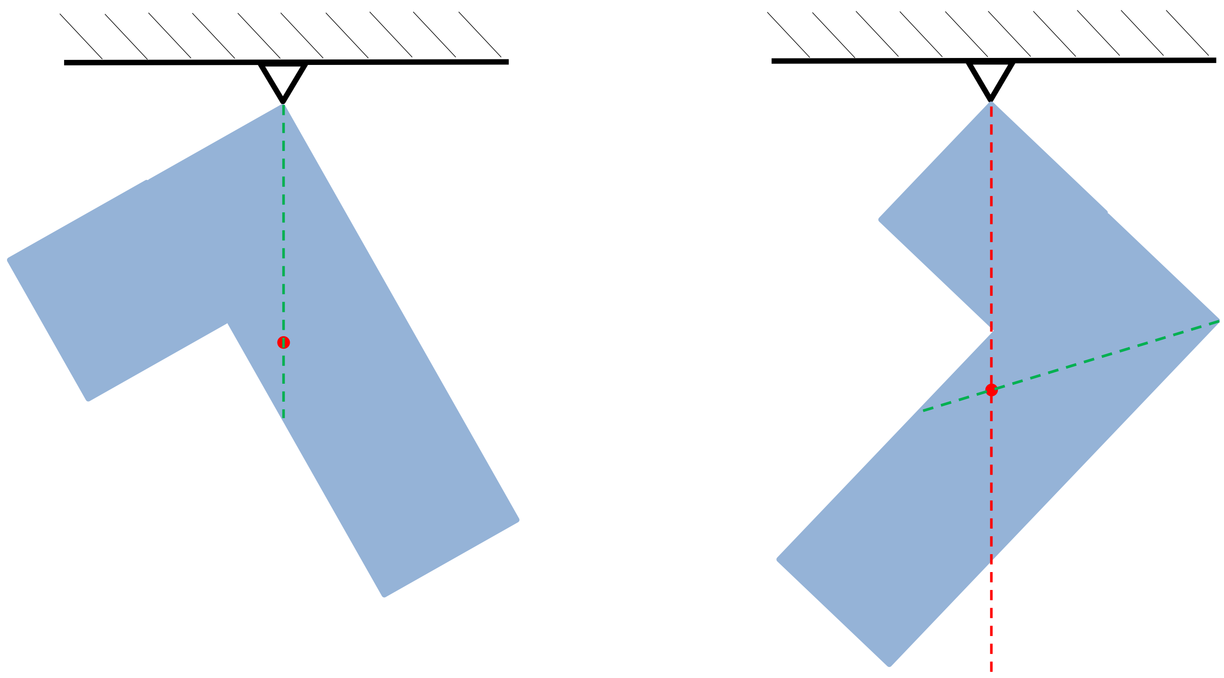
Find individual
CoMs & combine

<u>Complex shapes</u>



Integrate or
numerical integration

Finding the CoM experimentally



Finding the CoM experimentally

Plumb line method

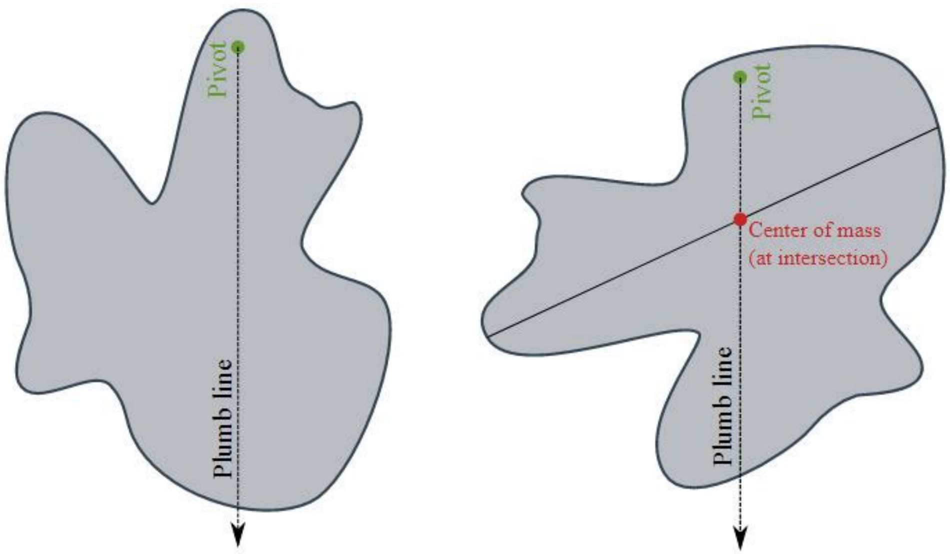
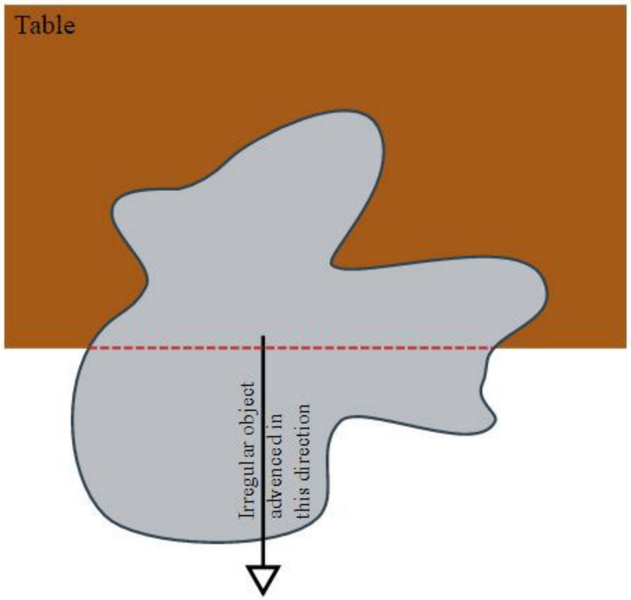


Table edge method



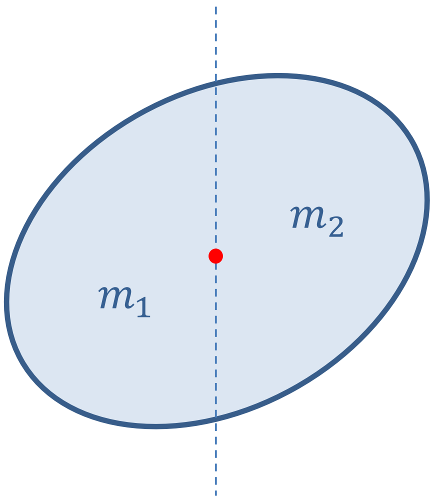
Example

The CoM is always inside the body?

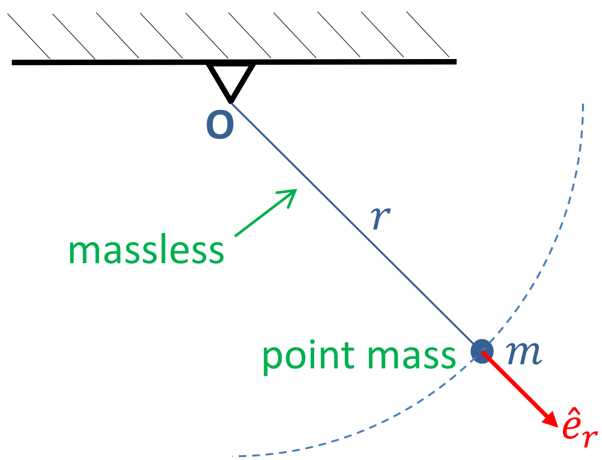
A. True
B. Fals
e

Example

Half of the mass of the body is to the left of CoM and half is to the right?



Mass Moments of Inertia (MOI) Unit is kgm^2



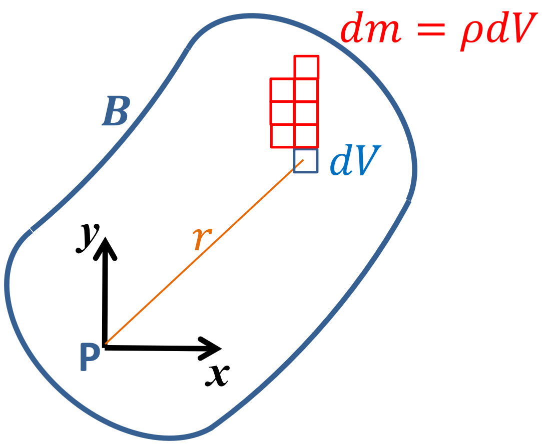
$I_{OZ} = mr^2$ for point mass

axis of rotation O-Z, O point of rotation

I_{OZ} is the resistance to rotation about the O-Z axis

r is perpendicular distance from axis of rotation to the point mass

$I_{O\hat{e}_r} = 0$

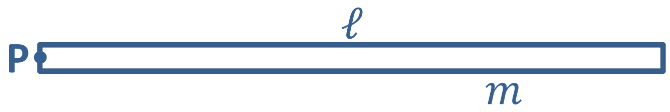


$$I_{PZ} = \iiint_B \rho r^2 dV \qquad dV = dxdydz$$

$$r^2 = x^2 + y^2$$

r is the orthogonal distance from the P-Z axis

MOI Tables

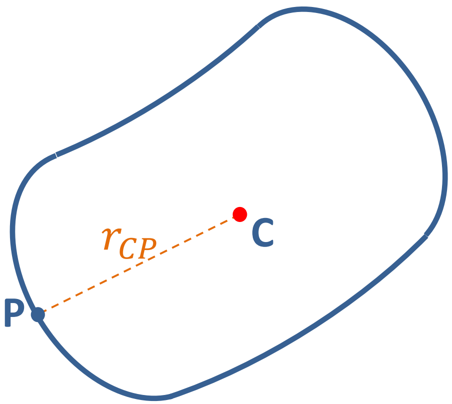


$$I_{PZ} = \frac{1}{3} m\ell^2$$ for thin rod

$$I_{PZ} = \iiint\limits_B \rho r^2 dV$$

Consider a broomstick rotated about one end.
If we cut the broomstick length in half, how much I_{PZ} does change?

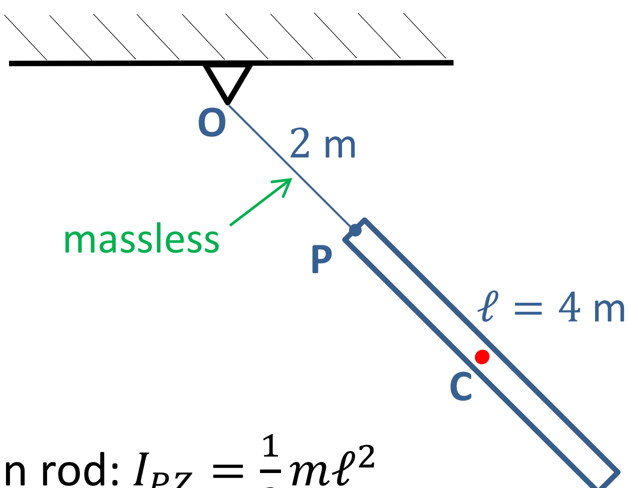
Parallel Axis Theorem



$$I_{PZ} = I_{CZ} + mr_{CP}^2$$

C = COM
P = other point
r_{CP} = distance orthogonal to Z-axis

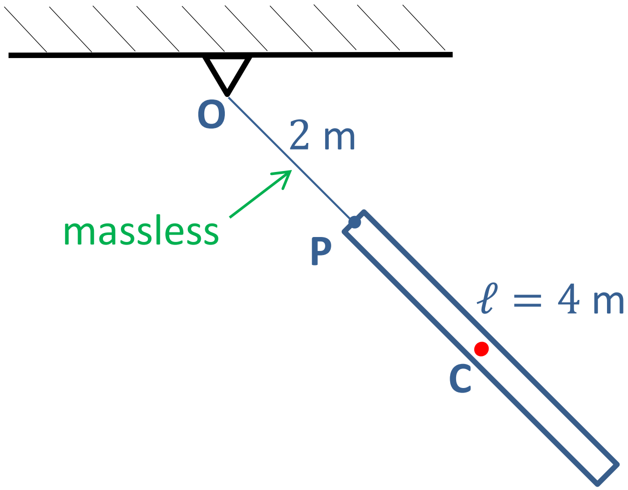
Example



Thin rod: $I_{PZ} = \frac{1}{3}m\ell^2$

$I_{PZ} = 16$
kgm^2

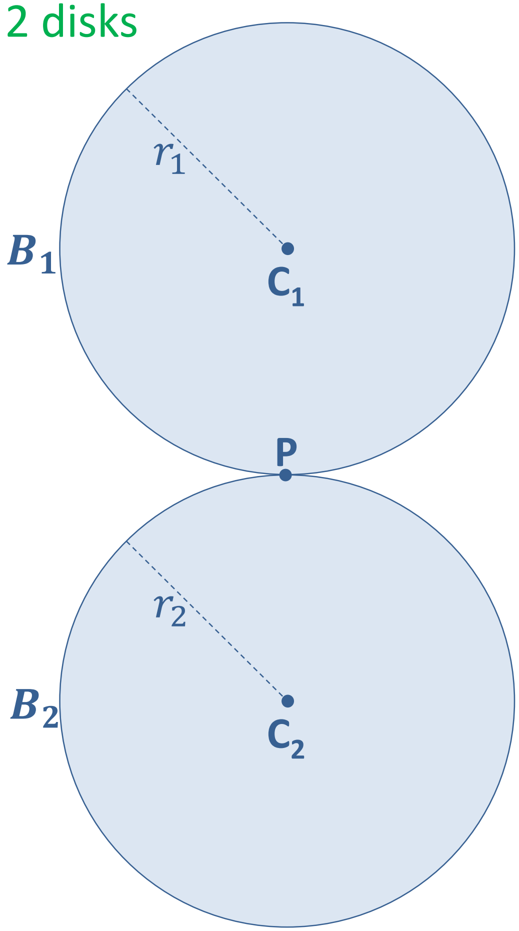
Example



$I_{PZ} = 16$
kgm^2

Additive Theorem

2 disks



$m_1 = 1 \text{ kg}$

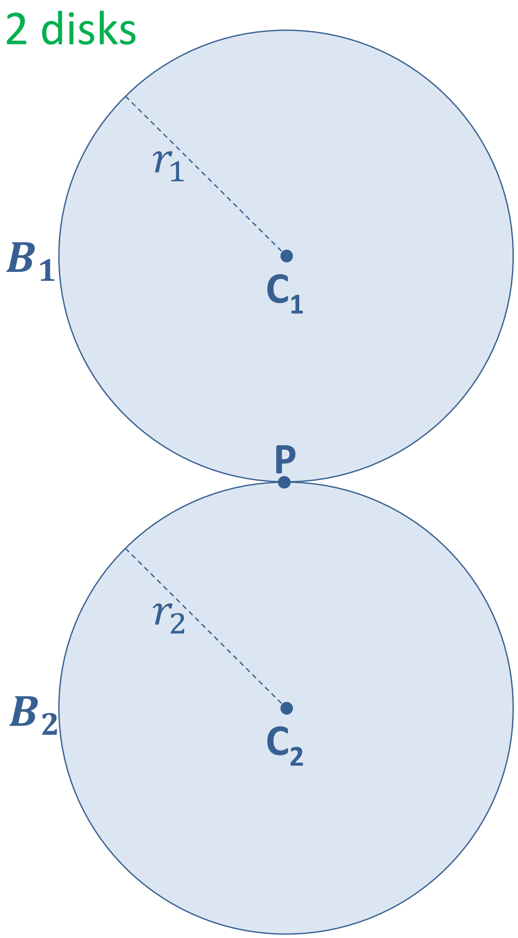
$r_1 = 2 \text{ m}$

$m_2 = 1 \text{ kg}$

$r_2 = 2 \text{ m}$

Disk: $I_{CZ} = \frac{1}{2} m r^2$

Additive Theorem



Parallel Axis Theorem

$$I_{PZ}^{B_1} = I_{C_1Z}^{B_1} + m_1 r_{PC_1}^2$$

$$I_{PZ}^{B_2} = I_{C_2Z}^{B_2} + m_2 r_{PC_2}^2$$

Additive Theorem

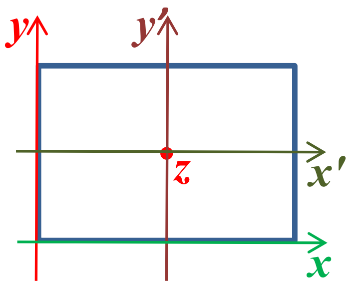
$$I_{PZ}^{total} = I_{PZ}^{B_1} + I_{PZ}^{B_2}$$

must be on the <u>same point</u>, along the <u>same axis</u> to be added together

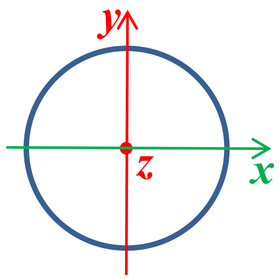
1. Determine location of P
2. Calculate individual MOI about individual centers
3. Shift individual MOI to P with Parallel Axis Theorem
4. Apply Additive Theorem

Finding MOI Theoretically

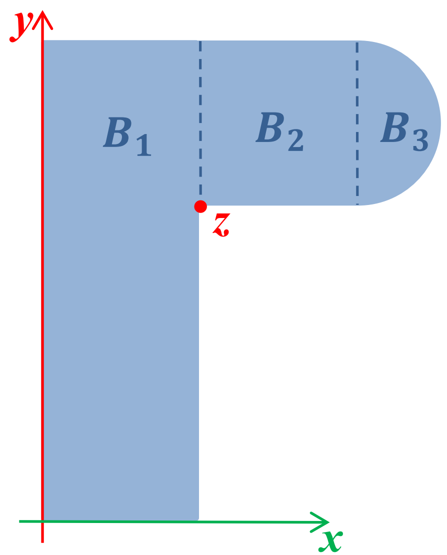
Simple shapes



Use tables

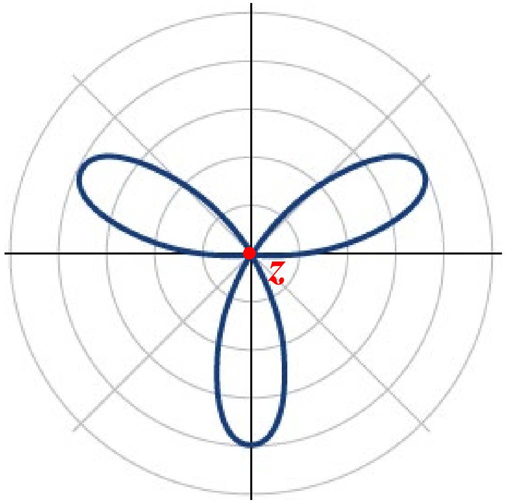


Composite shapes



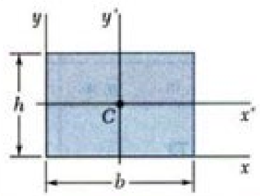
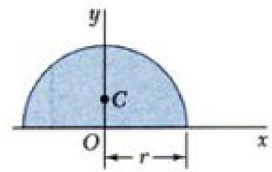
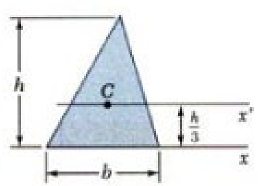
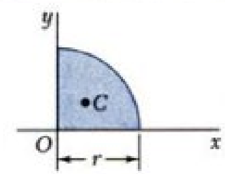
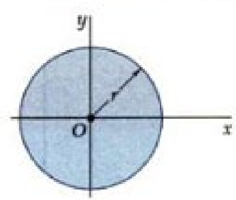
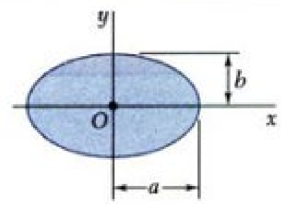
Parallel &
Additive
Theorems

Complex shapes



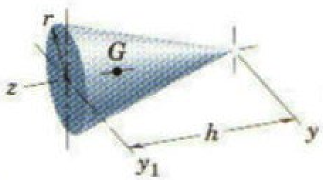
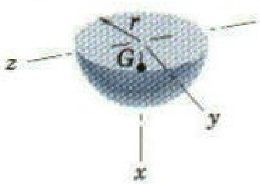
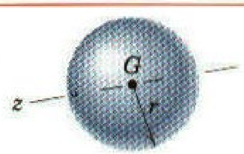
Integrate or
numerical integration

Finding MOI Theoretically

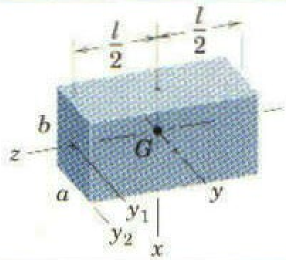
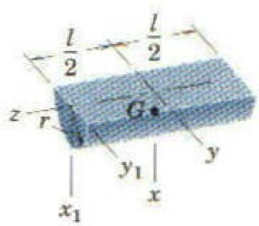
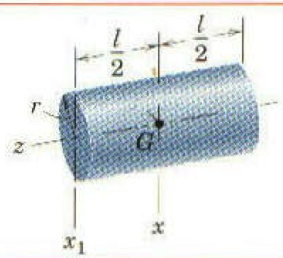
Rectangle		$\bar{I}_{x'} = \frac{1}{12}bh^3$ $\bar{I}_{y'} = \frac{1}{12}b^3h$ $I_x = \frac{1}{3}bh^3$ $I_y = \frac{1}{3}b^3h$ $J_C = \frac{1}{12}bh(b^2 + h^2)$	Semicircle		$I_x = I_y = \frac{1}{8}\pi r^4$ $J_O = \frac{1}{4}\pi r^4$
Triangle		$\bar{I}_{x'} = \frac{1}{36}bh^3$ $I_x = \frac{1}{12}bh^3$	Quarter circle		$I_x = I_y = \frac{1}{16}\pi r^4$ $J_O = \frac{1}{8}\pi r^4$
Circle		$\bar{I}_x = \bar{I}_y = \frac{1}{4}\pi r^4$ $J_O = \frac{1}{2}\pi r^4$	Ellipse		$\bar{I}_x = \frac{1}{4}\pi ab^3$ $\bar{I}_y = \frac{1}{4}\pi a^3b$ $J_O = \frac{1}{4}\pi ab(a^2 + b^2)$

This TAM 210 table is for <u>"Area" Moments of Inertia</u>

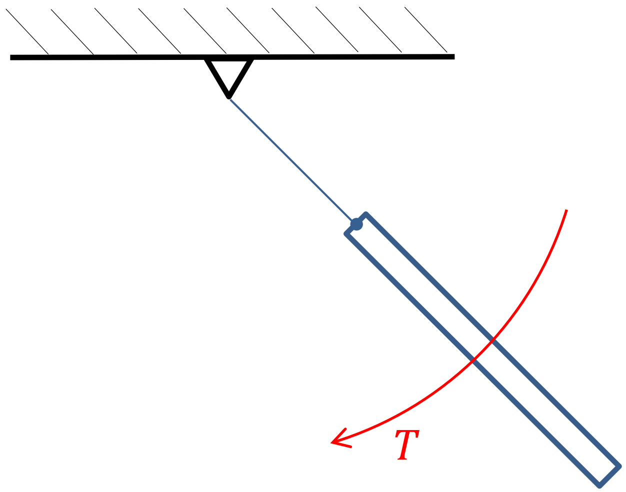
Finding MOI Theoretically

	Right-Circular Cone	$\bar{z} = \frac{3h}{4}$	$I_{yy} = \frac{3}{20}mr^2 + \frac{3}{5}mh^2$ $I_{y_1y_1} = \frac{3}{20}mr^2 + \frac{1}{10}mh^2$ $I_{zz} = \frac{3}{10}mr^2$ $\bar{I}_{yy} = \frac{3}{20}mr^2 + \frac{3}{80}mh^2$
	Hemisphere	$\bar{x} = \frac{3r}{8}$	$I_{xx} = I_{yy} = I_{zz} = \frac{2}{5}mr^2$ $\bar{I}_{yy} = \bar{I}_{zz} = \frac{83}{320}mr^2$
	Sphere	—	$I_{zz} = \frac{2}{5}mr^2$

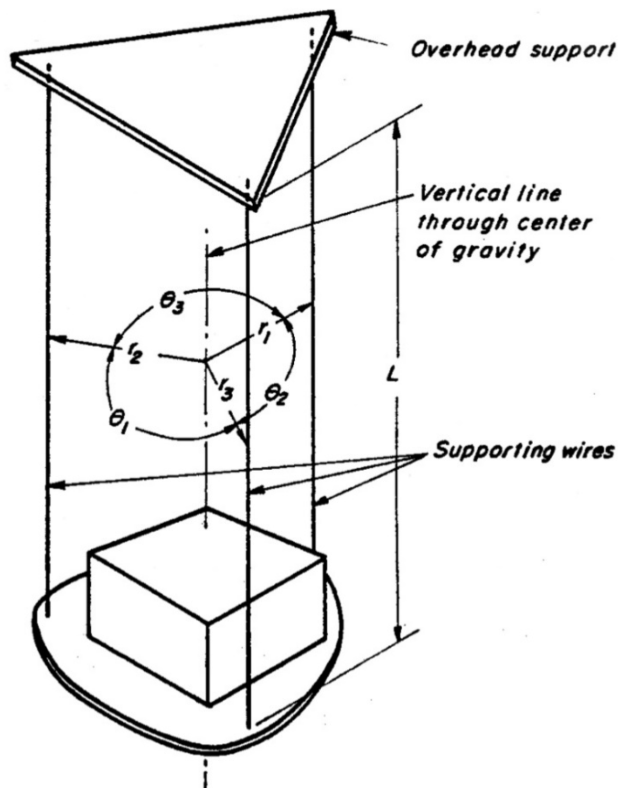
Finding MOI Theoretically

	Rectangular Parallelepiped	—	$I_{xx} = \frac{1}{12}m(a^2 + l^2)$ $I_{yy} = \frac{1}{12}m(b^2 + l^2)$ $I_{zz} = \frac{1}{12}m(a^2 + b^2)$ $I_{y_1y_1} = \frac{1}{12}mb^2 + \frac{1}{3}ml^2$ $I_{y_2y_2} = \frac{1}{3}m(b^2 + l^2)$
	Semicylinder	$\bar{x} = \frac{4r}{3\pi}$	$I_{xx} = I_{yy}$ $= \frac{1}{4}mr^2 + \frac{1}{12}ml^2$ $I_{x_1x_1} = I_{y_1y_1}$ $= \frac{1}{4}mr^2 + \frac{1}{3}ml^2$ $I_{zz} = \frac{1}{2}mr^2$ $\bar{I}_{zz} = \left(\frac{1}{2} - \frac{16}{9\pi^2}\right)mr^2$
	Circular Cylinder	—	$I_{xx} = \frac{1}{4}mr^2 + \frac{1}{12}ml^2$ $I_{x_1x_1} = \frac{1}{4}mr^2 + \frac{1}{3}ml^2$ $I_{zz} = \frac{1}{2}mr^2$

Finding the MOI experimentally

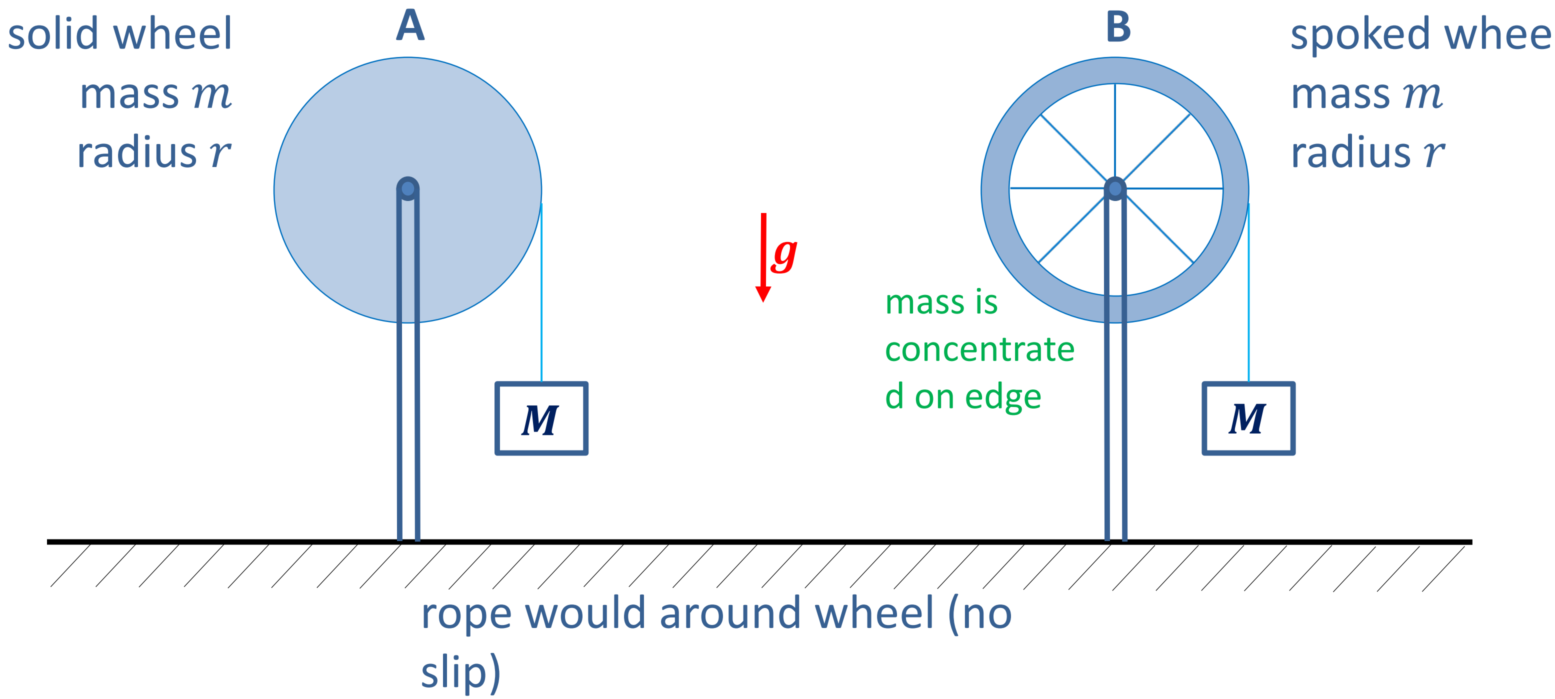


$T =$ period of oscillation
Can be used to determine MOI



Trifilar Pendulum designed by Charles Crede in 1948 has been used to measure period of oscillation in order to determine MOI.

Example


solid wheel
mass m
radius r
A
M
g
B
spoked wheel
mass m
radius r
mass is concentrated on edge
M
rope would around wheel (no slip)