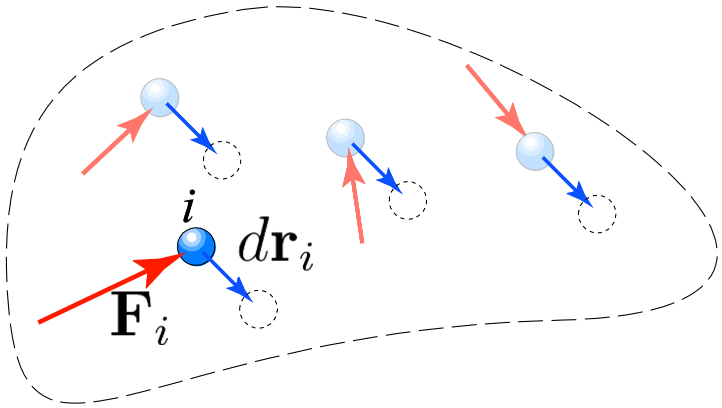
Chapter 11: Virtual Work

Goals and Objectives

- Introduce the principle of virtual work
- Show how it applies to determining the equilibrium configuration of a series of pin-connected members

Definition of Work

Work of a force



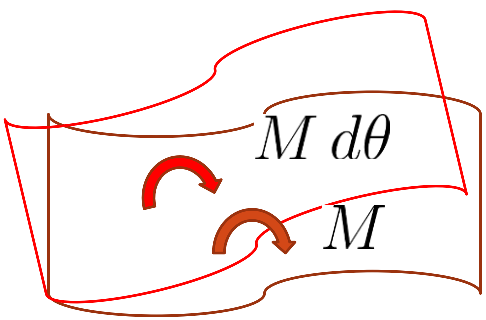
A force does work when it undergoes a displacement in the direction of the line of action.

The work dU produced by the force $\boldsymbol{F}$ when it undergoes a differential displacement $d\boldsymbol{r}$ is given by

$$dU = \mathbf{F} \cdot d\mathbf{r}$$

Work of a couple moment

$$dU = M\mathbf{k} \cdot d\theta\,\mathbf{k} = M\,d\theta$$

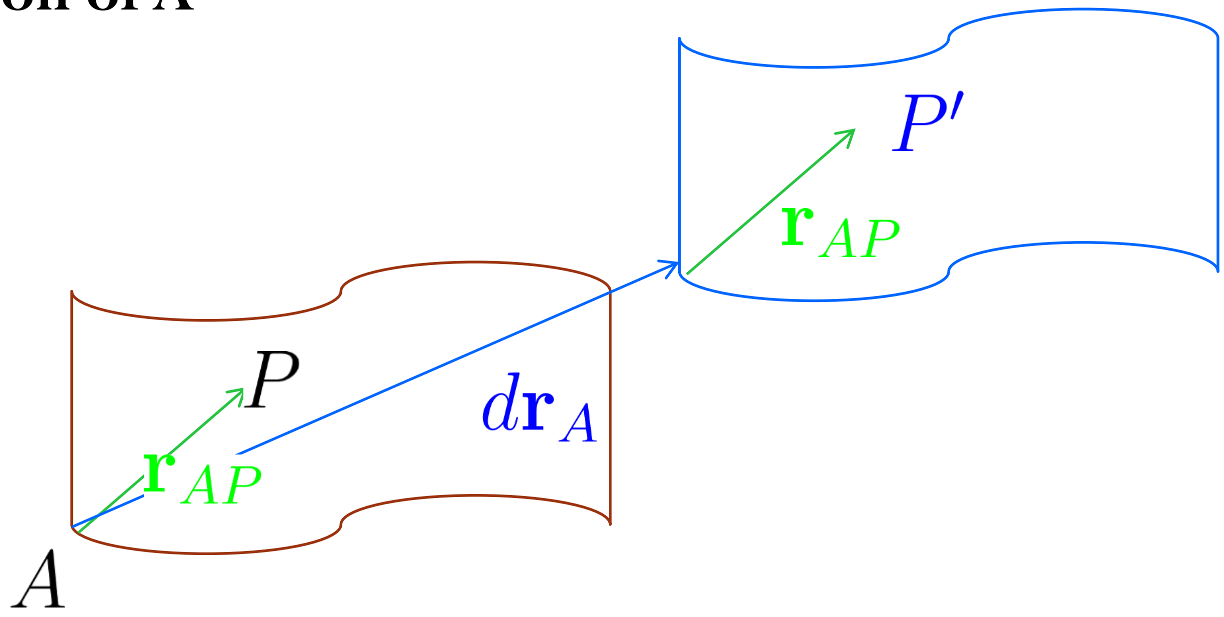


Incremental Displacement

Rigid body displacement of P = translation of A + rotation about A

$$d\mathbf{r}_P = d\mathbf{r}_A + d\theta\,\mathbf{k} \times \mathbf{r}_{AP}$$

Translation of A

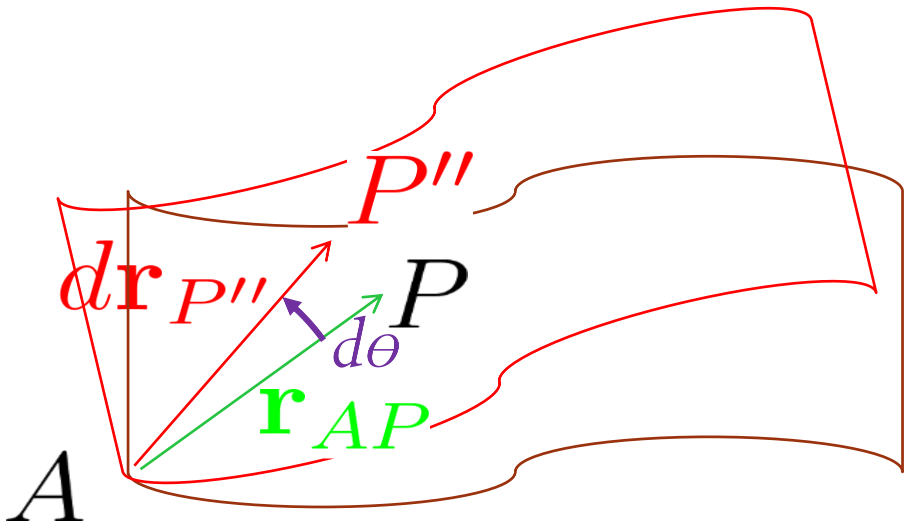


Incremental Displacement

Rigid body displacement of P = translation of A + rotation about A

$$d\mathbf{r}_P = d\mathbf{r}_A + \overbrace{d\theta\,\mathbf{k} \times \mathbf{r}_{AP}}^{d\mathbf{r}_{P''}}$$

Rotation about A



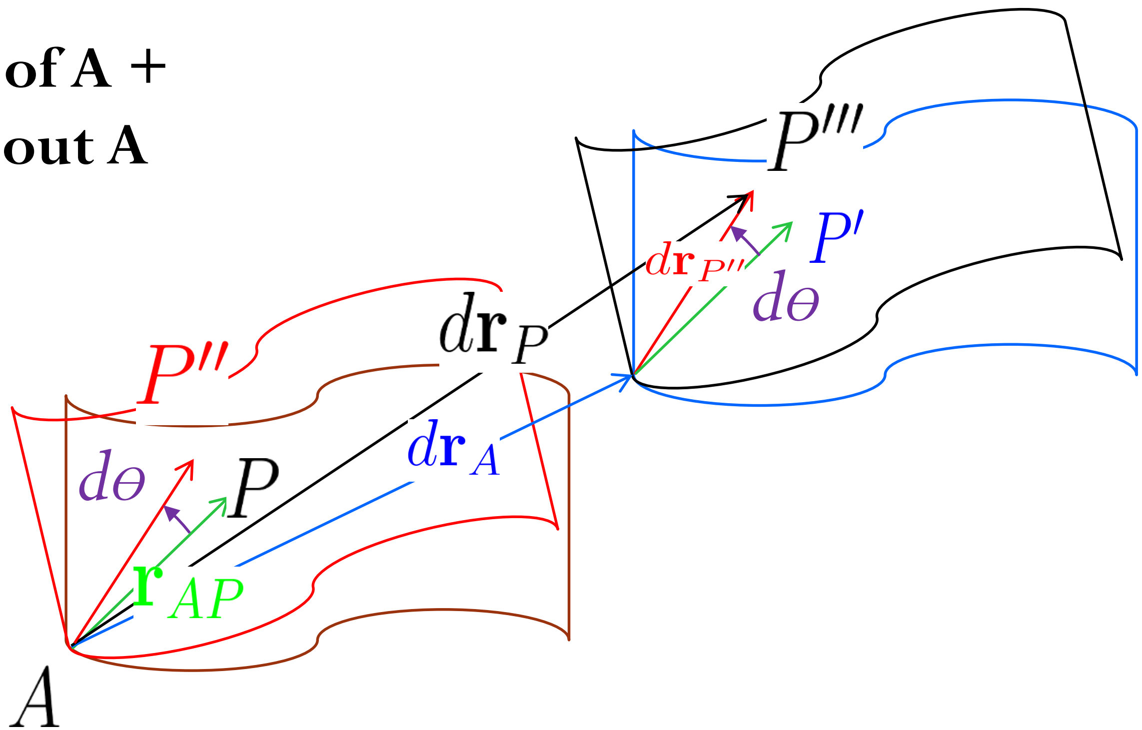
Incremental Displacement

Rigid body displacement of P = translation of A + rotation about A

$$d\mathbf{r}_P = d\mathbf{r}_A + d\theta\,\mathbf{k} \times \mathbf{r}_{AP}$$

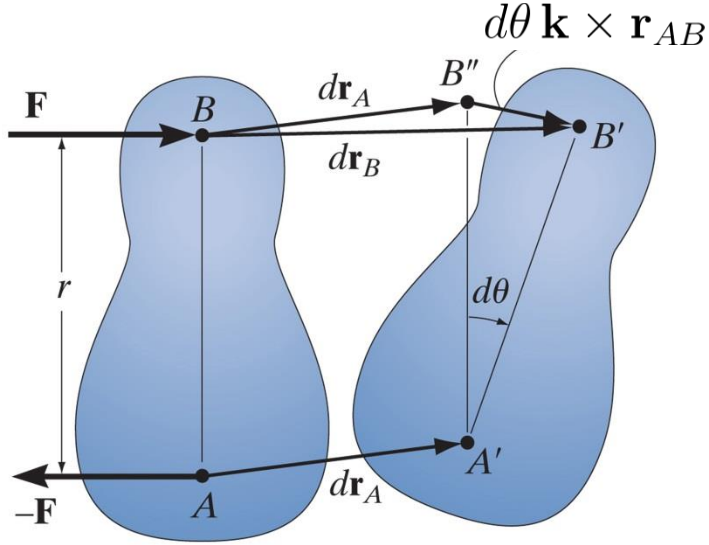
Translation of A +

Rotation about A



Definition of Work

Work of couple



$$d\mathbf{r}_P = d\mathbf{r}_A + d\theta\,\mathbf{k}\times\mathbf{r}_{AP}$$

$$\begin{aligned}
dU &= \sum_i \mathbf{F}_i\cdot d\mathbf{r}_i \\
&= \mathbf{F}_A\cdot d\mathbf{r}_A + \mathbf{F}_B\cdot d\mathbf{r}_B \\
&= -\mathbf{F}\cdot(d\mathbf{r}_A + d\theta\,\mathbf{k}\times\mathbf{r}_{AA}) + \mathbf{F}\cdot(d\mathbf{r}_A + d\theta\,\mathbf{k}\times\mathbf{r}_{AB}) \\
&= \mathbf{F}\cdot(d\theta\,\mathbf{k}\times\mathbf{r}_{AB}) \\
&= d\theta\,\mathbf{k}\cdot(\mathbf{r}_{AB}\times\mathbf{F}) \\
&= d\theta\,\mathbf{k}\cdot\mathbf{M}
\end{aligned}$$

$$\therefore\quad dU = M\mathbf{k}\cdot d\theta\,\mathbf{k} = M\,d\theta$$

The couple forces do no work during the translation $d\boldsymbol{r}_A$

Work due to rotation

Virtual Displacements

A *virtual displacement* is a conceptually possible displacement *or* rotation of all *or* part of a system of particles. The movement is assumed to be possible, but actually does not exist.

A virtual displacement is a first-order differential quantity denoted by the symbol δ (for example, δr and $\delta\theta$.

Principle of Virtual Work

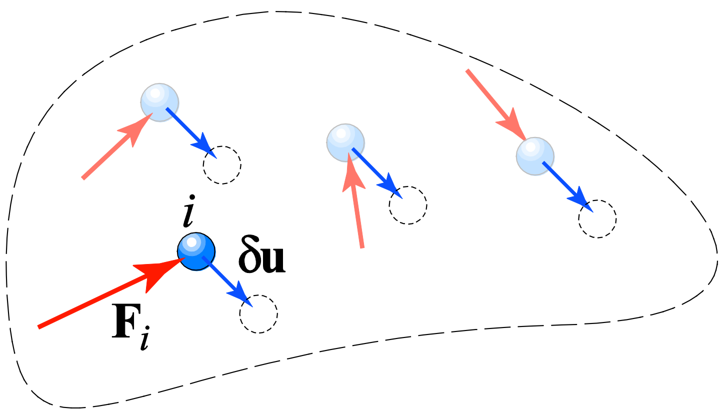
The principle of virtual work states that if a body is in equilibrium, then the algebraic sum of the virtual work done by all the forces and couple moments acting on the body is zero for any virtual displacement of the body. Thus,

$$\delta U = 0$$

$$\delta U = \Sigma(\boldsymbol{F} \cdot \delta\boldsymbol{u}) + \Sigma(\boldsymbol{M} \cdot \delta\boldsymbol{\theta}) = 0$$

For 2D:

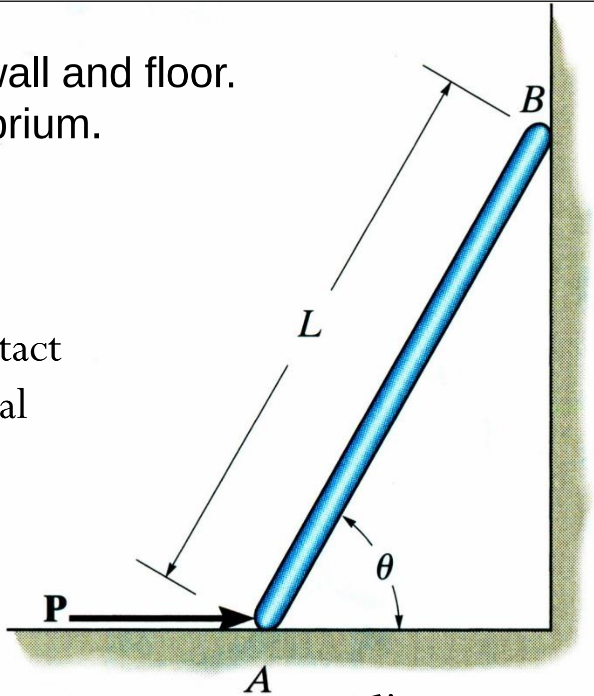
$$\delta U = \Sigma(\boldsymbol{F} \cdot \delta\boldsymbol{u}) + \Sigma(M\ \delta\theta) = 0$$



Procedure for Analysis

1. Draw FBD of the entire system and provide coordinate system
2. Sketch the "deflected position" of the system
3. Define position coordinates measured from a fixed point and select the parallel line of action component and remove forces that do no work
4. Differentiate position coordinates to obtain virtual displacement
5. Write the virtual work equation and express the virtual work of each force/ couple moment
6. Factor out the comment virtual displacement term and solve

Example: The thin rod of weight *W* rests against the smooth wall and floor. Determine the magnitude of force *P* needed to hold it in equilibrium.



Use the principle of virtual work. This problem has one degree of freedom, which we can take as the angle θ. Let $\delta\theta$ be the virtual rotation of the rod, such that the rod slides at A and B. Since the contact at A and B are smooth, the only forces that do work during the virtual displacements are P and W. Then the virtual work becomes:

$$\delta U = \boldsymbol{P}\cdot\delta\boldsymbol{u}_P + \boldsymbol{W}\cdot\delta\boldsymbol{u}_W = 0$$

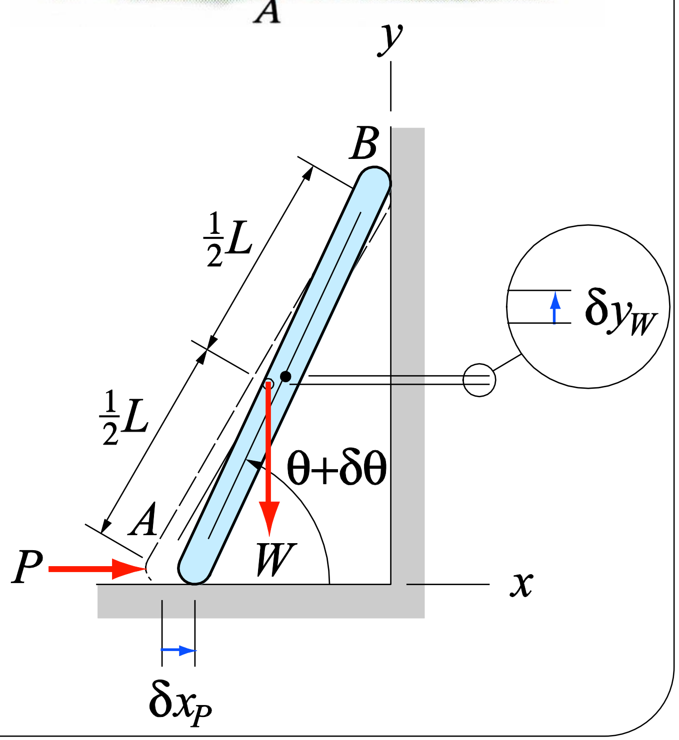
$$P\hat{\boldsymbol{\imath}}\cdot\delta x_P\,\hat{\boldsymbol{\imath}} - W\hat{\boldsymbol{\jmath}}\cdot(\delta x_W\,\hat{\boldsymbol{\imath}} + \delta y_W\hat{\boldsymbol{\jmath}}) = 0$$

$$P\,\delta x_P - W\,\delta y_W = 0 \;\Longrightarrow\; P = W\frac{\delta y_W}{\delta x_P}$$

$$x_P = -L\cos\theta \;\Longrightarrow\; \delta x_P = L\sin\theta\,\delta\theta$$

$$y_P = \frac{L}{2}\sin\theta \;\Longrightarrow\; \delta y_P = \frac{L}{2}\cos\theta\,\delta\theta$$

Hence, $$P = \frac{W}{2\tan\theta}$$



The effort required to solve this problem by the principle of virtual work is about the same as that by the equations of equilibrium for a free-body diagram of the rod.