Announcements

- CBTF Quiz 6 this week
- No class on Friday ☺ (But Friday discussions still meet)

❑ Upcoming deadlines:

- Wednesday (11/15)
  - PL HW22
- Thursday (11/16)
  - ME HW23

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Chapter 10: Moments of Inertia

Applications

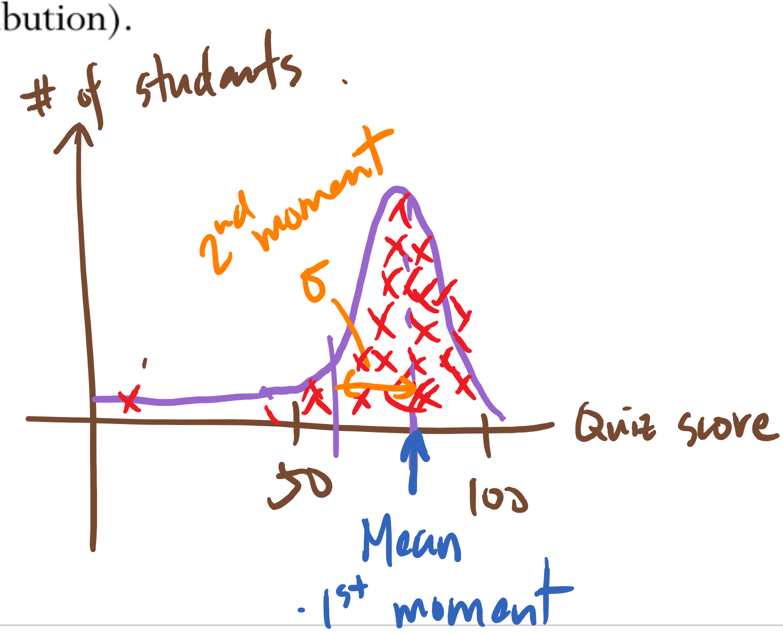
Many structural members like beams and columns have cross sectional shapes like an I, H, C, etc..

Why do they usually not have solid rectangular, square, or circular cross sectional areas?

What primary property of these members influences design decisions?

Terminology: the term **moment** in this module refers to the mathematical sense of different "measures" of an area or volume.

- The *zeroth* moment is the total mass, area, volume.
- The *first* moment (a single power of position) gave us the centroid.
- The *second* moment will allow us to describe the "width."
- An analogy that may help: in *probability* the first moment gives you the mean (the center of the distribution), and the second is the standard deviation (the width of the distribution).



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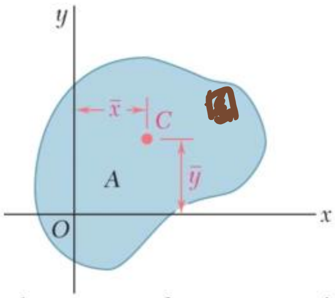
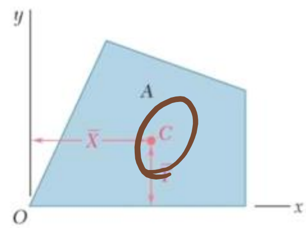
Recap: First moment of an area (centroid of an area)

- The first moment of the area A with respect to the x-axis is given by $Q_x = \int_A y\, dA$
- The first moment of the area A with respect to the y-axis is given by $Q_y = \int_A x\, dA$
- The centroid of the area A is defined as the point C of coordinates and , which satisfies the relation



element

$$\begin{cases} \int_A x\, dA = A\,\bar{x} \\ \int_A y\, dA = A\,\bar{y} \end{cases}$$



- In the case of a composite area, we divide the area A into parts

$$A_{total}\,\bar{X} = \sum_i A_i\,\bar{x}_i \qquad A_{total}\,\bar{Y} = \sum_i A_i\,\bar{y}_i$$

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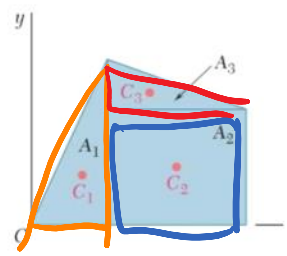

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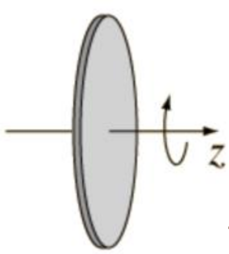
$$\bar{X} = \frac{\sum \tilde{x}_i A_i}{A_{total}}$$

Mass Moment of Inertia

(TA212 / TAM 251)

- **Mass moment of inertia** is the mass property of a rigid body that determines the torque needed for a desired angular acceleration about an axis of rotation.
- A larger mass moment of inertia around a given axis requires more torque to increase the rotation, or to stop the rotation, of a body about that axis
- Mass moment of inertia depends on the shape and density of the body and is different around different axes of rotation.

Torque-acceleration relation:

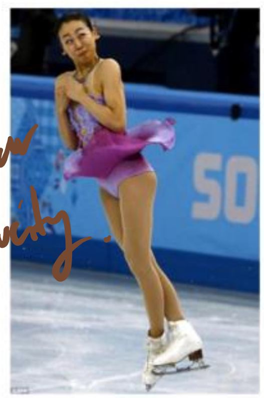


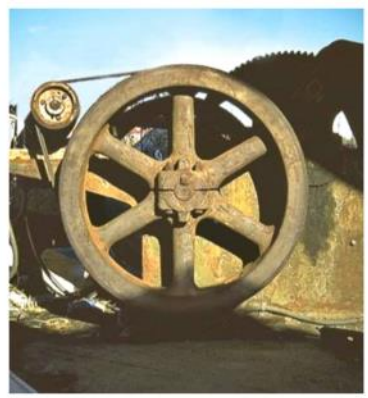
Torque

angular velocity

$$T = I\alpha$$

mass moment of inertia





Second moment of area

Moment of inertia is the property of a deformable body that determines the moment needed to obtain a desired curvature about an axis.
Moment of inertia depends on the shape of the body and may be different around different axes of rotation.

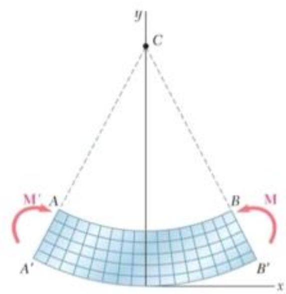
- The moment of inertia of the area A with respect to the x-axis is given by

$$I_x = \int y^2 dA.$$

- The moment of inertia of the area A with respect to the y-axis is given by

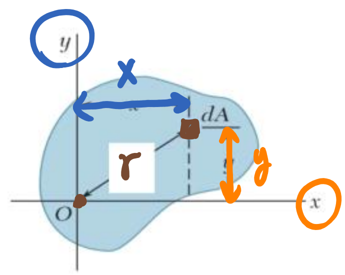
$$I_y = \int x^2 dA$$

- The moment of inertia of the area A with respect to the origin is given by (Polar MoI)



3 internal loadings:

N, V, M



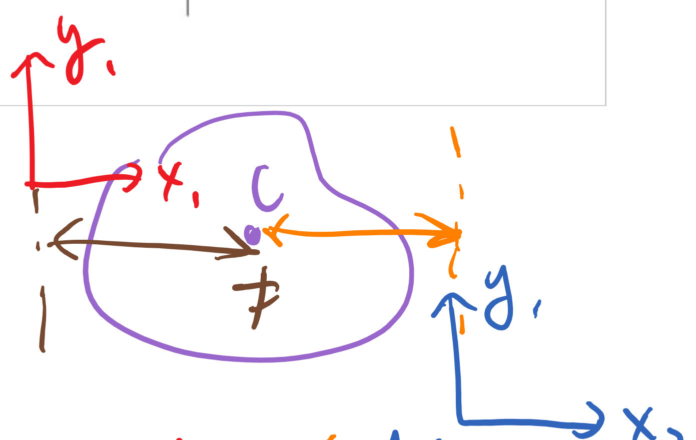
$$I_0 = \int r^2 dA.$$

$$r^2 = x^2 + y^2.$$

$$\Rightarrow I_0 = \int (x^2 + y^2) dA.$$

$$= \underbrace{\int x^2 dA}_{} + \underbrace{\int y^2 dA}_{}$$

$$= I_x + I_y.$$



$MoI_1 \neq MoI_2$

~ Dependent on reference axis.

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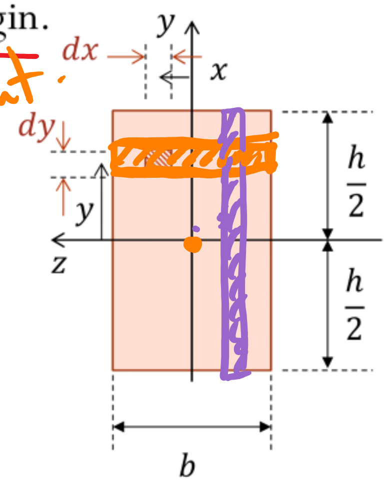
Moment of inertia of a rectangular area about the origin.



constant.

$$I_o = I_x + I_y.$$

① $I_x = \int y^2 dA$ $\quad dA = b\,dy$

$$= \int_{-\frac{h}{2}}^{\frac{h}{2}} y^2 (b)\, dy = \frac{by^3}{3}\Big|_{-\frac{h}{2}}^{h/2}$$

$$= \frac{b}{3}\left[\left(\frac{h}{2}\right)^3 - \left(-\frac{h}{2}\right)^3\right] = \frac{bh^3}{12}$$

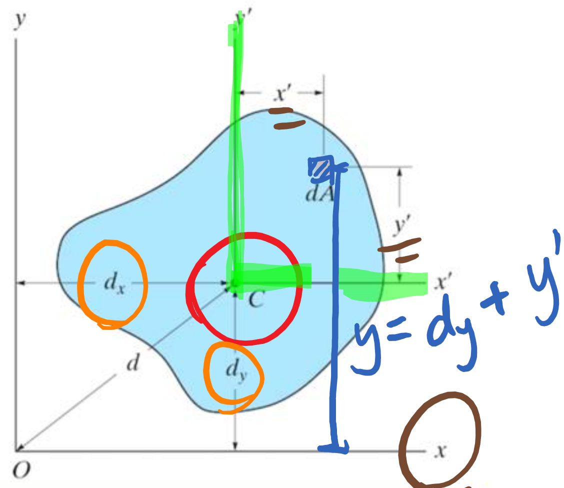
② $I_y = \int x^2 dA$ $\quad dA = h\,dx$

$$= \frac{hb^3}{12}$$

$$\Longrightarrow \quad I_o = \frac{bh^3}{12} + \frac{b^3h}{12}$$

Parallel axis theorem

- Often, the **moment of inertia** of an area is known for an axis passing through the **centroid**; e.g., x' and y':
- The moments around other axes can be computed from the known $I_{x'}$ and $I_{y'}$:



- First, express y (from x-axis to dA) in terms of d_y and y'

$$\bar{y} = d_y + y'$$

- Substitute into the moment of inertia equation: $I_x = \int y^2 dA$

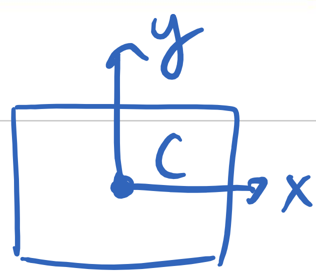
Note: the integral over y' gives zero *when done through the centroid axis.*

$$I_x = \int (y' + d_y)^2 dA.$$

$$= \int (y'^2 + 2y'd_y + d_y^2)\, dA.$$

$$= \int y'^2 dA + \int 2y' d_y dA + \int d_y^2 dA.$$

$$= I_{x'} + 0 + d_y^2 \int dA.$$

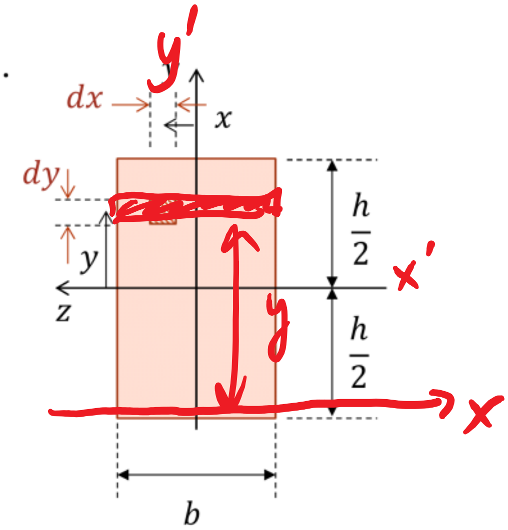


$$\frac{\int x\, dA}{\int dA} = 0$$

$$\boxed{I_x = I_{x'} + d_y^2 A. \qquad I_y = I_{y'} + d_x^2 A}$$

→ MoI (I_x, I_y) in terms of centroidal MoI ($I_{x'}$, $I_{y'}$) & d_x and d_y.

Moment of inertia of a rectangular area about its base.



- From the table

$$I_{x'} = \frac{bh^3}{12} \qquad I_{y'} = \frac{hb^3}{12}.$$

Parallel Axis Theorem

$$I_x = I_{x'} + d_y^2 A = \frac{bh^3}{12} + \left(\frac{h}{2}\right)^2 (bh)$$

$$= \frac{bh^3}{12} + \frac{bh^3}{4} = \frac{4bh^3}{12} = \frac{bh^3}{3}. \checkmark$$

- check solution using integral method.

$$I_x = \int y^2 dA \qquad dA = b\,dy$$

$$= \int_0^h y^2 (b) dy$$

$$= \frac{by^3}{3}\Big|_0^h = \frac{bh^3}{3} \checkmark$$

Same !! ☺

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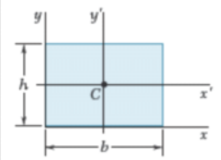
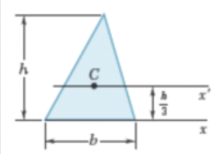
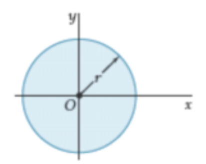
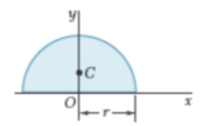
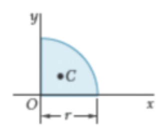
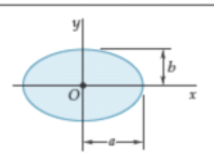
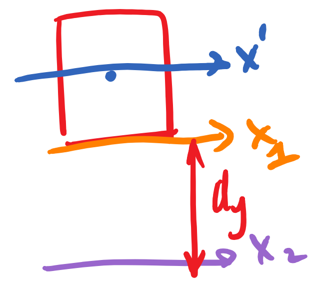
Rectangle		$\bar{I}_{x'} = \frac{1}{12}bh^3$ $\bar{I}_{y'} = \frac{1}{12}b^3h$ $I_x = \frac{1}{3}bh^3$ $I_y = \frac{1}{3}b^3h$ $J_C = \frac{1}{12}bh(b^2 + h^2)$
Triangle		$\bar{I}_{x'} = \frac{1}{36}bh^3$ $I_x = \frac{1}{12}bh^3$
Circle		$\bar{I}_x = \bar{I}_y = \frac{1}{4}\pi r^4$ $J_O = \frac{1}{2}\pi r^4$
Semicircle		$I_x = I_y = \frac{1}{8}\pi r^4$ $J_O = \frac{1}{4}\pi r^4$
Quarter circle		$I_x = I_y = \frac{1}{16}\pi r^4$ $J_O = \frac{1}{8}\pi r^4$
Ellipse		$\bar{I}_x = \frac{1}{4}\pi ab^3$ $\bar{I}_y = \frac{1}{4}\pi a^3b$ $J_O = \frac{1}{4}\pi ab(a^2 + b^2)$

) Not for Parallel Axis Theorem.

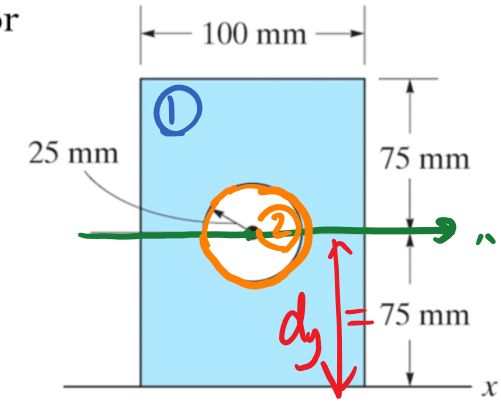
$I_{x_2} \neq I_{x_1} + d_y^2 A$



Moment of inertia of composite

- If individual bodies making up a **composite** body have individual areas *A* and moments of inertia *I* computed through their centroids, then the **composite area** and **moment of inertia** is a sum of the individual component contributions. $\boxed{I_x = \sum I_{x_i}}$
- This requires the **parallel axis theorem**
- Remember:
  - The position of the centroid of each component **must** be defined with respect to the same **same origin**.
  - It is allowed to consider **negative areas** in these expressions. Negative areas correspond to holes/missing area. **This is the one occasion to have negative moment of inertia**.

Determine the moment of inertia for the shaded area about the x-axis.



from here (centroid)

to heare

	①	②
$I_{x'}$	$\frac{bh^3}{12}$	$\frac{1}{4}\pi r^4$
d_y	75 mm	75 mm
A	15000 mm^2	$\pi(25)^2$ mm

Use parallel axis theorem:

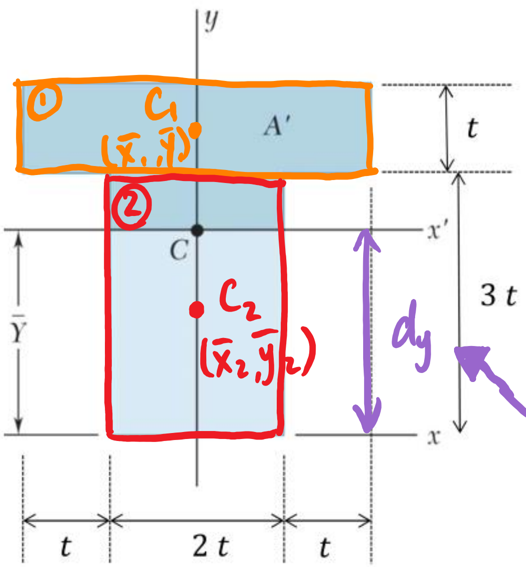
$$I_x = I_{x'} + d_y^2 A$$

$$\Rightarrow I_x = I_{x_1'} + d_y^2 A_1 - \left(I_{x_2'} + d_y^2 A_2\right)$$

$$= \left[\frac{(100mm)(150mm)^3}{12} + (75mm)^2(100mm)(150mm)\right] - \left[\frac{1}{4}\pi(25mm)^4 + (75mm)^2(\pi)(25mm)^2\right]$$

$$\underline{I_x \approx 1.01 \times 10^8 mm^4}$$

Find the moment of inertia about its centroid:



From last chapter: Centroid position of the area below is given by

$\bar{y}_1 = 3.5t$ $\quad A_{total}\,\bar{Y} = \sum_i A_i\,\bar{y}_i \quad$ $\bar{y}_2 = 1.5t$

$A_1 = (4t)t$ $\quad A_2 = (2t)(3t)$

$$\bar{Y} = \frac{\underbrace{4t^2\,(3.5t)}_{A_1\bar{y}_1} + \underbrace{6t^2\,(1.5t)}_{A_2(\bar{y}_2)}}{\underbrace{4t^2 + 6t^2}_{A_1 + A_2}} = \boxed{\frac{23t}{10} = \bar{Y}}$$

• Use symmetry: $\boxed{\bar{x} = 0}$

Can you use this d_y to find $I_{x'}$?

$I_{x'} = I_x + d_y^2 A$