Majorana Neutrinos

We next discuss the topics of Majorana fields and Majorana mass

We start with the Dirac equation:

$$(i\gamma^\mu \partial_\mu - m)\psi = 0 \quad , \quad \psi = \psi_L + \psi_R$$

By apply $P_L = \frac{1}{2}(1-\gamma^5)$ and $P_R = \frac{1}{2}(1+\gamma^5)$ from the left, we obtain

$$i\gamma^\mu \partial_\mu \psi_L = m\psi_R$$
$$i\gamma^\mu \partial_\mu \psi_R = m\psi_L$$

where $\psi_R = \frac{1}{2}(1+\gamma^5)\psi$; $\psi_L = \frac{1}{2}(1-\gamma^5)\psi$

If $m=0$, then

$$i\gamma^\mu \partial_\mu \psi_L = 0 \quad ; \quad i\gamma^\mu \partial_\mu \psi_R = 0$$

These are the Weyl equations, and ψ_L and ψ_R are the Weyl spinors.

* ψ_L and ψ_R each have only two independent components, even though each contains 4 components (Recall $\psi_L = \begin{pmatrix} 0 \\ \chi_L \end{pmatrix}$ and $\psi_R = \begin{pmatrix} \chi_R \\ 0 \end{pmatrix}$ in the Chiral representation)

Neutral current

$$\bar{u}_f \gamma_\mu (C_V^f - C_A^f \gamma^5) u_f$$

f : fermion

$$C_V^f = T_f^3 - 2 \sin^2\theta_W Q_f$$

$$C_A^f = T_f^3$$

f	C_A^f	C_V^f
ν_e	$\frac{1}{2}$	$\frac{1}{2}$
e^-	$-\frac{1}{2}$	$-\frac{1}{2} + 2\sin^2\theta_W$

$$g_L^f = \frac{1}{2}(C_V^f + C_A^f)$$

$$g_R^f = \frac{1}{2}(C_V^f - C_A^f)$$

f	g_L^f	g_R^f
ν_e	$\frac{1}{2}$	0
e^-	$-\frac{1}{2} + \sin^2\theta_W$	$\sin^2\theta_W$

* A massless fermion can be described by a single chiral field, either ψ_L or ψ_R, as suggested by Weyl equations. However, Pauli rejected this idea using parity conservation.

Under parity transformation

$$i\gamma^\mu \partial_\mu \psi_L = 0 \quad \text{becomes} \quad i\gamma^\mu \partial_\mu \psi_R = 0$$

hence, both ψ_L and ψ_R are required if parity is conserved.

* The discovery of parity violation in 1957, and the indication from experiments that neutrinos are massless, suggest that neutrinos could be described by the Weyl equation afterall. Indeed, in the Standard Model, we only have ν_L, and ν_R does not exist.

* We now know that ν is massive. Therefore, the Weyl equation does not apply to neutrinos. However, is it necessary to abandon the concept of two-component Weyl spinors? In orderwords, is it nessary to use 4-component Dirac spinors to describe neutrinos?

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* The answer to the above question is NO!
One can still use two component weyl spinors,
the only consequence is that ψ_L, ψ_R are no
longer independent! This was discovered by
Majorana in 1937.

* Majorana suggested the possibility of a
"Majorana equation". The idea is to express
ψ_R in

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$$i\gamma^\mu \partial_\mu \psi_L = m\psi_R$$

in terms of ψ_L

In particle, one can express ψ_R as

$$\psi_R = (\psi_L)^c = \xi C \overline{\psi_L}^T = C\overline{\psi_L}^T \quad (\text{set } \xi = 1)$$

This implies that ψ_R is not independent
of ψ_L.

Since ψ_R is a right-handed field, we need to
show that $C\overline{\psi_L}^T$ is also right-handed:

$$P_L\{C\overline{\psi_L}^T\} = [\frac{1-\gamma^5}{2}] C\overline{\psi_L}^T = C\{\frac{1-(\gamma^5)^T}{2}\}\overline{\psi_L}^T$$

((since $C\gamma_\mu^T C^{-1} = -\gamma_\mu$, we have $C(\gamma^5)^T C^{-1} = \gamma^5$
and $\gamma^5 C = C(\gamma^5)^T$)

Hence $P_L[C\overline{\psi_L}^T] = C[\overline{\psi_L}(\frac{1-\gamma^5}{2})]^T = C\{\overline{\psi}(\frac{1+\gamma^5}{2})(\frac{1-\gamma^5}{2})\}^T = 0$

We now briefly review how Dirac spinors transform under

1) Lorentz transformation
2) Parity
3) charge conjugation

1) Lorentz transformation

$$(i \gamma^{\mu} \frac{\partial}{\partial x^{\mu}} - m)\, \psi(x) = 0 \qquad \text{------ (a)}$$

Lorentz transformation:

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\nu}$$

The covariance of Dirac Eq. under the Lorentz transformation implies

$$(i \gamma^{\mu} \frac{\partial}{\partial x'^{\mu}} - m)\, \psi'(x') = 0 \qquad \text{--- (b)}$$

The Dirac spinors transform as

$$\psi'(x') = \psi'(\Lambda x) = S\, \psi(x) \qquad \text{--- (c)}$$

(note that for a scalar field, $\psi'(x') = \psi(x)$, but this is not so for a spin $= 1/2$ fermion field, or a spin $= 1$ vector field)

From Eq. (c), we have $\psi(x) = S^{-1} \psi'(x')$ --- (d)

Eq. (d) and Eq. (a) and the expression

$$\frac{\partial}{\partial x^{\mu}} = \frac{\partial}{\partial x'^{\nu}} \frac{\partial x'^{\nu}}{\partial x^{\mu}} = \Lambda^{\nu}{}_{\mu} \frac{\partial}{\partial x'^{\nu}}$$

gives

$$\Lambda^{\nu}{}_{\mu}\, \gamma^{\mu} = S^{-1} \gamma^{\nu} S$$

when Eq. (c) is used.

For parity trans. $\vec{x} \to -\vec{x}$, $t \to t$

$$\Lambda = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$$\gamma^0 = S_p^{-1} \gamma^0 S_p$$

$$-\gamma^i = S_p^{-1} \gamma^i S_p$$

$$[S_p, \gamma^0] = 0 \quad , \quad \{S_p, \gamma^i\} = 0$$

$$S_p = \gamma^0$$

* Another way to obtain this result

$$(i\gamma^\mu \frac{\partial}{\partial x^\mu} - m)\psi(x) = 0$$

$$\gamma^0 (i\gamma^\mu \frac{\partial}{\partial x^\mu} - m)\psi(x) = 0$$

$$(i\gamma^\mu \frac{\partial}{\partial x'^\mu} - m)\gamma^0 \psi(x) = 0$$

$x'^0 = x^0$, $x'^i = -x^i$ (parity trans.)

$$(i\gamma^\mu \frac{\partial}{\partial x'^\mu} - m)\psi'(x') = 0$$

$\psi'(x') = \gamma^0 \psi(x)$ under parity

For a Lorentz boost or rotation

$$S = \exp\left[-\frac{i}{4} \omega_{\mu\nu} \sigma^{\mu\nu}\right]$$

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$$

antisymmetric tensor

$$g^{\mu\nu} = \frac{1}{2}\{\gamma^\mu, \gamma^\nu\}$$

symmetric tensor

$$S_{Rot} = \exp\left[\frac{i}{2} \vec{\omega} \cdot \vec{\Sigma}\right] ; \quad \Sigma_i = \begin{bmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{bmatrix}$$

$$S_{Boost} = \exp\left[-\frac{1}{2} \vec{\omega} \cdot \vec{\alpha}\right] ; \quad \alpha_i = \begin{bmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{bmatrix}$$

For Charge-conjugation tranformation, the free Dirac Eq. does not tell us anything, since it is not sensitive to the charge of the ~~fermion~~ e^-/e^+

$$\partial^\mu \to D^\mu \equiv \partial^\mu + iqA^\mu$$

$$[\gamma^\mu(i\partial_\mu + eA_\mu) - m]\psi = 0$$

$$[\gamma^\mu(i\partial_\mu - eA_\mu) - m]\psi_c = 0$$

$$\psi_c = S_c \psi^* \quad ; \quad S_c (\gamma^\mu)^* S_c^{-1} = -\gamma^\mu$$

S_c commutes with γ^2, anticommutes with $\gamma^0, \gamma^1, \gamma^3$

$$S_c \equiv i\gamma^2 = \begin{bmatrix} 0 & & & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & & 0 \end{bmatrix}$$

(inverting the upper and lower components)

$$\psi_c = i\gamma^2 \psi^*$$

or $\psi_c = C\bar{\psi}^T \qquad C = i\gamma^2\gamma^0$

How does ψ_c transform under Lorentz tranf.?

$$\psi \to \exp\left[-\frac{i}{4} w_{\mu\nu} \sigma^{\mu\nu}\right] \psi$$

$$\psi^* \longrightarrow \exp\left\{\frac{i}{4} w_{\mu\nu} (\sigma^{\mu\nu})^*\right] \psi^*$$

$$i\gamma^2\psi^* \longrightarrow i\gamma^2 \exp\left\{\frac{i}{4} w_{\mu\nu} (\sigma^{\mu\nu})^*\right] \psi^*$$

$$= \exp\left[-\frac{i}{4} w_{\mu\nu} \sigma^{\mu\nu}\right] i\gamma^2\psi^*$$

Hence $\psi_c \Longrightarrow \exp\left\{-\frac{i}{4} w_{\mu\nu} \sigma^{\mu\nu}\right] \psi_c$

under Lorentz tranf

$i\not{\partial}\psi = m\psi$ Dirac Eq.

$i\not{\partial}\psi = m\psi_c$ Majorana Eq.

Therefore $C\bar{\psi}_L^T$, or ψ_L^c, is right-handed.

$i\gamma^\mu \partial_\mu \psi_L = m\psi_R$ becomes $i\gamma^\mu \partial_\mu \psi_L = m\psi_L^c$

and

$i\gamma^\mu \partial_\mu \psi_R = m\psi_L$ becomes $i\gamma^\mu \partial_\mu \psi_L^c = m\psi_L$

Note that $i\gamma^\mu \partial_\mu \psi_L^c = m\psi_L$ is obtained from $i\gamma^\mu \partial_\mu \psi_L = m\psi_L^c$ by taking charge-conjugation operation.

* For Majorana field

$$\psi = \psi_L + \psi_R = \psi_L + C\bar{\psi}_L^T = \psi_L + \psi_L^c$$

It is clear that, $\psi^c = C(\bar{\psi})^T = \psi_L^c + (\psi_L^c)^c$
$= \psi_L^c + \psi_L = \psi$

hence, $\psi = \psi^c$

as expected for a Majorana field.

One can show $(\psi^c)^c = \psi$ by using

$$\psi^c = C\bar{\psi}^T = -\gamma^0 C\psi^*$$

and $C\gamma_\mu^T C^{-1} = -\gamma_\mu$, $C^+ = C^{-1}$, $C^T = -C$

(or use $C = i\gamma^2\gamma^0$, which is true for both the Dirac and Chiral representations)

Since $j^\mu = q\bar{\psi}\gamma^\mu\psi$ changes sign under charge conjugation, and $\psi = \psi^c$ for a Majorana particle, one sees that $j^\mu = 0$ for a Majorana particle. One can also show this explicitly,

$$\bar{\psi}\gamma^\mu\psi = \overline{\psi^c}\gamma^\mu\psi^c \quad \text{for Majorana particle } (\psi = \psi^c)$$

$$\overline{\psi^c}\gamma^\mu\psi^c = -\psi^T C^\dagger \gamma^\mu C \bar{\psi}^T = [\ \ ]^T = \bar{\psi} C (\gamma^\mu)^T C^\dagger \psi$$

$$= -\bar{\psi}\gamma^\mu\psi = 0 .$$

* For massive Dirac neutrinos, the degree of freedom is 4, while for massive Majorana neutrinos, the degree of freedom is 2. To show this graphically:

Dirac

$$\begin{tikzcd}
\nu(\vec{p}, h) \arrow[r, leftrightarrow, "CPT"] \arrow[d, leftrightarrow, "\text{boost}"] & \bar{\nu}(\vec{p}, -h) \arrow[d, leftrightarrow, "\text{boost}"] \\
\nu(-\vec{p}, -h) \arrow[r, leftrightarrow, "CPT"] \arrow[d, leftrightarrow, "\text{rotation}"] & \bar{\nu}(-\vec{p}, h) \arrow[d, leftrightarrow, "\text{rotation}"] \\
\nu(\vec{p}, -h) & \bar{\nu}(\vec{p}, h)
\end{tikzcd}$$

Majorana

$$\begin{tikzcd}
\nu(\vec{p}, h) \arrow[r, leftrightarrow, "CPT"] \arrow[d, leftrightarrow, "\text{boost}"] & \nu(\vec{p}, -h) \arrow[d, leftrightarrow, "\text{boost}"] \\
\nu(-\vec{p}, -h) \arrow[r, leftrightarrow, "CPT"] \arrow[d, leftrightarrow, "\text{rotation}"] & \nu(-\vec{p}, h) \arrow[d, leftrightarrow, "\text{rotation}"] \\
\nu(\vec{p}, -h) & \nu(\vec{p}, h)
\end{tikzcd}$$