Chapter 2

Weak Interaction involving neutrinos

There are numerous examples of weak interactions and weak decays. They can be characterized as proceeding via neutral current (NC), charged current (CC), or charged current plus neutral current (NC + CC). Also, they can be classified according to whether they are purely leptonic, semi-leptonic, or non-leptonic. An incomplete list follows:

	Charged Current	Neutral Current	Charged/Neutral
Leptonic	$\mu \to e\nu\bar{\nu}$	$\nu_\mu e \to \nu_\mu e$	$\nu_e e \to \nu_e e$
	$\tau \to \ell\nu\bar{\nu}$	$e^+e^- \to \ell^+\ell^-$	$e^+e^- \to \nu_e\bar{\nu}_e$
	$\nu_\mu e \to \mu\nu e$		
Semi-leptonic	$\pi \to \mu\nu$	$\nu N \to \nu N$	
	$D \to K\ell\nu$	$\bar{\nu}_e + D \to \bar{\nu}_e + n + p$	
	$n \to pe^-\bar{\nu}$		
	$\nu_\mu N \to \mu^- x$		
Non-leptonic	$K \to \pi\pi$	$pp \to pp$	$pn \to pn$
	$D \to K\pi$		
	$\Lambda \to p\pi^-$		

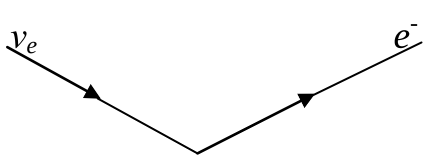
In purely leptonic processes, only leptons appear in the interactions or decays. For semi-leptonic processes, both hadrons and leptons participate. In non-leptonic processes, only hadrons appear.

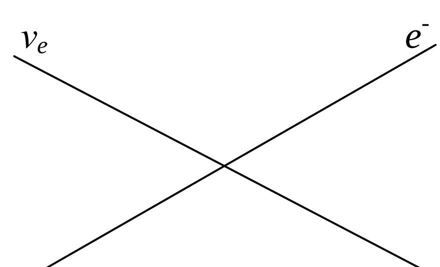
Pure Leptonic Weak Interaction

We consider the following reaction:

$$\nu_e e^- \to e^- \nu_e$$

This reaction can proceed via charged current as well as neutral current





This reaction was used to detect solar neutrino in several water Cherenkov detector experiments.

We now consider the charged-current contribution to this process. At low energy and intermediate energy, it is appropriate to adopt Fermi's contact current-current interaction. The invariant amplitude is

$$M = \frac{G}{\sqrt{2}}\left(\bar{u}(K')\gamma^{\mu}\left(1-\gamma^{5}\right)u(p)\right)\left(\bar{u}(p')\gamma_{\mu}\left(1-\gamma^{5}\right)u(K)\right) \tag{1}$$

To evaluate $|M|^2$, one needs M^* which contains adjoint current such as

$$\begin{aligned}\left[\bar{u}(K')\gamma^{\mu}\left(1-\gamma^{5}\right)u(p)\right]^{*} &= \left[\bar{u}(K')\gamma^{\mu}\left(1-\gamma^{5}\right)u(p)\right]^{\dagger} \\ &= \bar{u}(p)\gamma^{\mu}\left(1-\gamma^{5}\right)u(K')\end{aligned} \tag{2}$$

(Note that $\left[\bar{u}(K')\left(1-\gamma^{5}\right)u(p)\right]^{*} = \bar{u}(p)\left(1+\gamma^{5}\right)u(K')$)

$$\begin{aligned}\overline{|M|^{2}} = \frac{1}{2}\sum_{spin}|M|^{2} &= \frac{G^{2}}{4}Tr\left(\gamma^{\mu}\left(1-\gamma^{5}\right)\not{p}\gamma^{\nu}\left(1-\gamma^{5}\right)\not{K'}\right) \\ &\times Tr\left(\gamma_{\mu}\left(1-\gamma^{5}\right)\not{K}\gamma_{\nu}\left(1-\gamma^{5}\right)\not{p'}\right)\end{aligned} \tag{3}$$

Note that in Equation 3, a factor of ½ is used instead of ¼ for the average of the initial spin states since ν_e is left-handed and is in a unique spin state (the electron is unpolarized and can be in either spin states).

Noting $\quad Tr\left(\gamma^{\mu}\left(1-\gamma^{5}\right)\not{p}\gamma^{\nu}\left(1-\gamma^{5}\right)\not{K'}\right) = 2Tr\left(\gamma^{\mu}\not{p}\gamma^{\nu}\left(1-\gamma^{5}\right)\not{K'}\right)$

Equation 3 becomes

$$\overline{|M|^2} = G^2 Tr\left(\gamma^\mu \not{p} \gamma^\nu \left(1-\gamma^5\right) \not{K'}\right) Tr\left(\gamma_\mu \not{K} \gamma_\nu \left(1-\gamma^5\right) \not{p'}\right) \tag{4}$$

Some useful trace theorems are listed below:

$$Tr\left(\gamma^\mu \not{P}_1 \gamma^\nu \not{P}_2\right) = 4\left(P_1^\mu P_2^\nu + P_1^\nu P_2^\mu - \left(P_1 \bullet P_2\right) g^{\mu\nu}\right) \tag{5}$$

$$Tr\left(\gamma^\mu \not{P}_1 \gamma^\nu \gamma^5 \not{P}_2\right) = 4i\varepsilon^{\mu\alpha\nu\beta} P_{1\alpha} P_{2\beta} \tag{6}$$

($\varepsilon^{\mu\alpha\nu\beta}$ is antisymmetric tensor for $0 \le \varepsilon,\ \alpha,\ \gamma,\ \beta \le 3$, $\varepsilon^{0123} = -1$ and it changes sign upon permutation)

Equations 5 and 6 give

$$\begin{aligned} &Tr\left(\gamma^\mu \not{P}_1 \gamma^\nu \not{P}_2\right) Tr\left(\gamma_\mu \not{P}_3 \gamma_\nu \not{P}_4\right) \\ &= 32\left[\left(P_1 \bullet P_3\right)\left(P_2 \bullet P_4\right) + \left(P_1 \bullet P_4\right)\left(P_2 \bullet P_3\right)\right] \end{aligned} \tag{7}$$

and

$$\begin{aligned} &Tr\left(\gamma^\mu \not{P}_1 \gamma^\nu \gamma^5 \not{P}_2\right) Tr\left(\gamma_\mu \not{P}_3 \gamma_\nu \not{P}_4\right) \\ &= 32\left[\left(P_1 \bullet P_3\right)\left(P_2 \bullet P_4\right) - \left(P_1 \bullet P_4\right)\left(P_2 \bullet P_3\right)\right] \end{aligned} \tag{8}$$

(note that $\varepsilon^{\mu\nu\lambda\sigma} \varepsilon_{\mu\nu\kappa\tau} = -2\left(\delta_\kappa^\lambda \delta_\tau^\sigma - \delta_\tau^\lambda \delta_\kappa^\sigma\right)$)

Note that

$$Tr\left(\gamma^\mu \not{P}_1 \gamma^\nu \not{P}_2\right) Tr\left(\gamma_\mu \not{P}_3 \gamma_\nu \gamma^5 \not{P}_4\right) = 0 \tag{9}$$

symmetric WRT $\mu\nu$ interchange

antisymmetric WRT $\mu\nu$ interchange

using Equations 7, 8 and 9, Equation 4 becomes

$$\begin{aligned} \overline{|M|^2} &= 64 G^2 \left(K \bullet P\right)\left(K' \bullet P'\right) \\ &= 16 G^2 S^2 \qquad \left(\text{ignoring electron's mass}\right) \end{aligned} \tag{10}$$

In the C.M. frame, the differential cross-section is

$$\frac{d\sigma}{d\Omega}\left(\nu_e e^- \to e^- \nu_e\right) = \frac{1}{64\pi^2 S}\overline{|M|^2} = \frac{G^2 S}{4\pi^2} \tag{11}$$

The angular distribution is isotropic, and the total cross-section is

$$\sigma\left(\nu_e e^- \to e^- \nu_e\right) = \frac{G^2 S}{\pi} \tag{12}$$

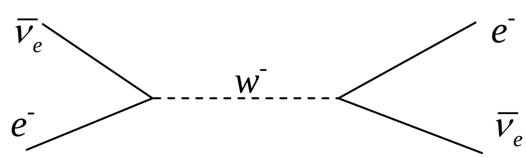
Note that in the lab frame, the angular distribution is no longer isotropic, due to the boost. Hence the $\nu_e e^- \to e^- \nu_e$ reaction can still be used to isolate ν_e originating from the sun.

The cross-sections for the $\nu_\mu e^- \to \mu^- \nu_e$ reaction, which can only proceed via charged current interaction, are identical to the CC part of $\nu_e e^- \to e^- \nu_e$ and are given by Equations 11 and 12.

We now consider another related reaction

$$\bar{\nu}_e e^- \to e^- \bar{\nu}_e$$

The reaction proceeds via an intermediate w boson



The $\bar{\nu}_e e^- \to e^- \bar{\nu}_e$ is related to $\nu_e e^- \to e^- \nu_e$ via crossing symmetry.

Interchanging ν_e's in the initial and final states in $\nu_e e^- \to e^- \nu_e$ would lead to $\bar{\nu}_e e^- \to e^- \bar{\nu}_e$. ($P_A \leftrightarrow -P_D$)

$$S' = \left(P'_A + P'_B\right)^2 = \left(-P_D + P_B\right)^2 = \left(P_C - P_A\right)^2 = t$$
$$t' = \left(P'_A - P'_C\right)^2 = \left(P_D + P_C\right)^2 = \left(P_A + P_B\right)^2 = S$$

Hence, $s \leftrightarrow t$ would relate $\nu_e e^- \to e^- \nu_e$ to $\bar{\nu}_e e^- \to e^- \bar{\nu}_e$. Interchanging $s \leftrightarrow t$, Equation 10 becomes

$$\overline{|M|^2} = 16G^2 t^2 = 4G^2 S^2 (1-\cos\theta)^2 \tag{13}$$

$$\text{(since } t = -\frac{S}{2}(1-\cos\theta)\text{)}$$

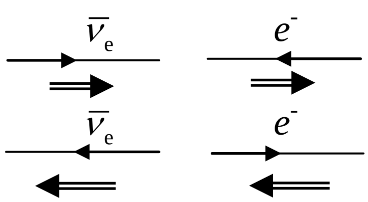
The differential cross-section in the C.M. frame is

$$\frac{d\sigma}{d\Omega}\left(\bar{\nu}_e e^- \to e^- \bar{\nu}_e\right) = \frac{G^2 S}{16\pi^2}(1-\cos\theta)^2 \tag{14}$$

and the total cross-section is

$$\sigma\left(\bar{\nu}_e e^- \to e^- \bar{\nu}_e\right) = \frac{G^2 S}{3\pi} = \frac{1}{3}\sigma\left(\nu_e e^- \to e^- \nu_e\right) \tag{15}$$

Equation 14 shows that the reaction is backward-peaked and the cross-section vanishes at $\theta = 0^o$. This can be readily understood from helicity consideration. In the C.M. frame



in the initial state (16)

in the final state for $\theta_{\text{C.M.}} = 0$

Angular momentum conservation prohibits scattering to $\theta_{\text{C.M.}} = 0$

An intuitive interpretation for the factor ⅓ appearing in Equation 15 is that the figure in 16 shows that only one of the three helicity states of *w* can participate in the $\bar{\nu}_e e^- \to e^- \bar{\nu}_e$ reaction.

Another reaction closely related to the $\nu_e e^- \to e^- \nu_e$ is

$$e^+ e^- \to \bar{\nu}_e \nu_e$$

This reaction can be obtained from $\nu_e e^- \rightarrow e^- \nu_e$ by crossing ν_e with e^-,
i.e. $P_A \leftrightarrow -P_C$

Therefore, $$S' = \left(P'_A + P'_B\right)^2 = \left(-P_C + P_B\right)^2 = \left(P_A - P_D\right)^2 = u$$

Interchanging $s \leftrightarrow u$ in Equation 10, we have

$$\overline{\left|M\right|^2} = 16G^2u^2 = 4G^2S^2\left(1+\cos\theta\right)^2 \tag{17}$$

$$\left(\text{since } u = \frac{-s}{2}\left(1+\cos\theta\right)\right)$$

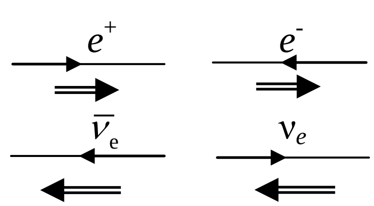
The differential cross-section is

$$\frac{d\sigma}{d\Omega}\left(e^+e^- \rightarrow \bar{\nu}_e\nu_e\right) = \frac{G^2 s}{16\pi^2}\left(1+\cos\theta\right)^2 \tag{18}$$

and the total cross-section is

$$\sigma\left(e^+e^- \rightarrow \bar{\nu}_e\nu_e\right) = \frac{G^2 s}{3\pi} = \sigma\left(\bar{\nu}_e e^- \rightarrow e^- \bar{\nu}_e\right) = \frac{1}{3}\sigma\left(\nu_e e^- \rightarrow e^- \nu_e\right) \tag{19}$$

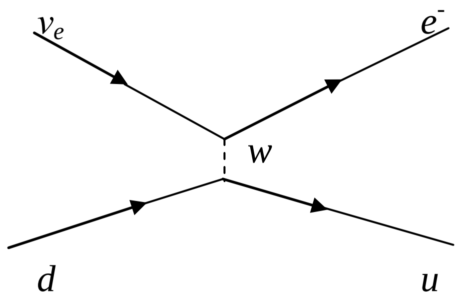
Equation 18 shows that $\bar{\nu}_e$ in the $e^+e^- \rightarrow \bar{\nu}_e\nu_e$ cannot go to 180^o. Again, this can be understood by considering angular momentum conservation:

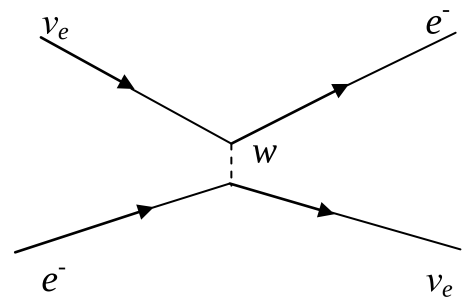


in the initial state

in the final state for $\theta_{C.M.} = 180^o$, and is not allowed due to angular momentum conservation

Another reaction closely related to $\nu_e e^- \to e^- \nu_e$ is the $\nu_e d \to e^- u$ reaction





The $\nu_e d \to e^- u$ is a semi-leptonic process, but the cross-section is almost identical to that of $\nu_e e^- \to e^- \nu_e$. The only difference is that $\nu_e d \to e^- u$ also contains the $\cos^2\theta_c$ term to account for the mixing between d and s.

Similarly, one can show that the $\nu_e u \to e^+ d$ reaction is the analog of $\bar{\nu}_e \nu_e \to e^+ e^-$ reaction.

We can summarize the above discussion with the following table. If one considers only the charged current and set the Cabbibo angle θ_c to 0, then

$\frac{d\sigma}{d\Omega} = \frac{G^2 s}{4\pi^2}$	$\frac{d\sigma}{d\Omega} = \frac{G^2 s}{16\pi^2}(1-\cos\theta)^2$	$\frac{d\sigma}{d\Omega} = \frac{G^2 s}{16\pi^2}(1+\cos\theta)^2$	
$\nu_e e^- \to e^- \nu_e$	$\bar{\nu}_e e^- \to e^- \bar{\nu}_e$	$e^+ e^- \to \bar{\nu}_e \nu_e$	
$\nu_\mu e^- \to \mu^- \nu_e$	$\bar{\nu}_e e^- \to \mu^- \bar{\nu}_\mu$	$\mu^+ e^- \to \bar{\nu}_\mu \nu_e$	
$\nu_e d \to e^- u$		$\bar{\nu}_e u \to e^+ d$	(20)
$\bar{\nu}_e \bar{d} \to e^+ \bar{u}$		$\nu_e \bar{u} \to e^- \bar{d}$	
$\nu_\mu d \to \mu^- u$		$\bar{\nu}_\mu u \to \mu^+ d$	
$\bar{\nu}_\mu \bar{d} \to \mu^+ \bar{u}$		$\nu_\mu \bar{u} \to \mu^- \bar{d}$	

Note that in (20), an isotropic angular distribution is obtained if the initial colliding pair have identical helicities (both are left-handed, or both are right-handed). If they have opposite helicity, then the cross-section is anisotropic and the integrated cross-section drops to ⅓ of the isotropic reactions.

The table (20) shows that there are no interactions between $\nu_\mu u,\ \nu_\mu \bar{d},\ \bar{\nu}_\mu d,\ \bar{\nu}_\mu \bar{u}$.

<u>Neutrino-Induced Deep-Inelastic Scattering (DIS)</u>

The underlying processes for neutrino induced DIS off a nucleon include:

$$\frac{d\sigma}{d\Omega}\left(\nu_\mu d \to \mu^- u\right) = \frac{G^2 s}{4\pi^2}$$
$$\frac{d\sigma}{d\Omega}\left(\bar{\nu}_\mu u \to \mu^+ d\right) = \frac{G^2 s}{16\pi^2}\left(1+\cos\theta\right)^2 \qquad (21)$$

It is useful to express Equation 21 in terms of Lorentz invariant quantities such as y, where $y = \frac{p \bullet q}{p \bullet K}$

$$y = \frac{p \bullet q}{p \bullet K} = \frac{p \bullet \left(K - K'\right)}{p \bullet K} = 1 - \frac{p \bullet K'}{p \bullet K} = 1 - \frac{1}{2}\left(1+\cos\theta\right) \qquad (22)$$

Hence, $1 + \cos\theta = 2(1 - y)$

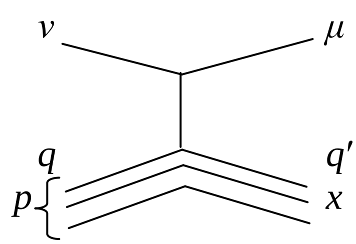
Also,
$$d\Omega = 2\pi \sin\theta d\theta = -2\pi d\cos\theta = 4\pi dy$$
$$\frac{d\sigma}{d\Omega} = \frac{1}{4\pi}\frac{d\sigma}{dy} \qquad (23)$$

Equation 21 should be written as

$$\begin{aligned}\frac{d\sigma}{d\Omega}\left(\nu_\mu d \to \mu^- u\right) &= \frac{G^2\hat{s}}{4\pi^2}\\ \frac{d\sigma}{d\Omega}\left(\overline{\nu}_\mu u \to \mu^+ d\right) &= \frac{G^2 s}{16\pi^2}\left(1+\cos\theta\right)^2\end{aligned} \tag{24}$$

where $\hat{s}$ represents the Mandelstam parameter s in the $\nu_\mu d$ system. Similarly, one can define $\hat{t}$ and $\hat{u}$.

For a DIS process



$\hat{s}, \hat{t}, \hat{u}$ refer to the $\nu + q \to \mu + q'$ subprocess, and s, t, u refer to the $\nu + N \to \mu + x$ process.

$$\hat{s} = \left(\hat{P}_a + \hat{P}_b\right)^2 \simeq 2\hat{P}_a \bullet \hat{P}_b = 2\left(P_a\right)\bullet\left(xP_b\right) = x\bullet 2P_a \bullet P_b = xs \tag{25}$$

Similarly, one can show that $\quad \hat{t} = t;\ \hat{u} = xu$

Using Equations 22, 23 and 25, Equation 24 becomes

$$\begin{aligned}\frac{d\sigma}{dy}\left(\nu_\mu d \to \mu^- u\right) &= \frac{G^2 xs}{\pi} \qquad \text{(same for } \overline{\nu}_\mu \overline{d} \to \mu^+ \overline{u}\text{)}\\ \frac{d\sigma}{dy}\left(\overline{\nu}_\mu u \to \mu^+ d\right) &= \frac{G^2 xs}{\pi}\left(1-y\right)^2 \qquad \text{(same for } \nu_\mu \overline{u} \to \mu^- \overline{d}\text{)}\end{aligned} \tag{26}$$

Equation 26 corresponds to DIS on the quark which carries a fraction x of the nucleon's momentum. For DIS on a nucleon, one needs to take into account the probability that the quark carries a momentum fraction x. Hence

$$\frac{d\sigma}{dxdy}\left(\nu_\mu p \to \mu^- x\right) = \frac{G^2 xs}{\pi}\left[d_p(x) + (1-y)^2\,\overline{u}_p(x)\right] \tag{27}$$

where the scattering off an antiquark is also considered.

(Note that in Equation 27 there is no e_q^2 factor, since it is a weak interaction and the coupling is not proportional to e^2.)

Similarly, for a DIS off a neutron, we have

$$\frac{d\sigma}{dxdy}\left(\nu_\mu n \to \mu^+ x\right) = \frac{G^2 xs}{\pi}\left[d_n(x) + (1-y)^2\,\overline{u}_n(x)\right] \tag{28}$$

Isospin symmetry demands $d_n(x) = u_p(x)$; $\overline{u}_n(x) = \overline{d}_p(x)$

For a scattering off an isoscalar target, which has an equal number of protons and neutrons (like d, ^{12}C, ^{40}Ca, . . .), the DIS cross-section per nucleon is an average of Equations 27 and 28:

$$\frac{d\sigma}{dxdy}\left(\nu_\mu N \to \mu^- x\right) = \frac{G^2 xs}{2\pi}\left[\left(u_p(x) + d_p(x)\right) + (1-y)^2\left(\overline{u}_p(x) + \overline{d}_p(x)\right)\right] \tag{29}$$

Similarly,

$$\frac{d\sigma}{dxdy}\left(\overline{\nu}_\mu N \to \mu^+ x\right) = \frac{G^2 xs}{2\pi}\left[\left(\overline{u}_p(x) + d_p(x)\right) + (1-y)^2\left(\overline{u}_p(x) + \overline{d}_p(x)\right)\right] \tag{30}$$

Note that the $\overline{\nu}_\mu N$ DIS is obtained from the $\nu_\mu N$ DIS by $s \leftrightarrow u$ interchange (or $1 \leftrightarrow (1-y)^2$ interchange).

Equations 29 and 30 show that a comparison between

$$\frac{d\sigma}{dxdy}\left(\nu_\mu N \to \mu^- x\right)$$

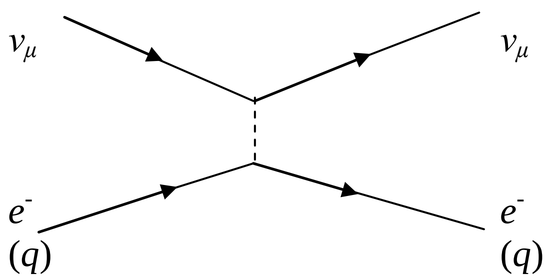
and

$$\frac{d\sigma}{dxdy}\left(\bar{\nu}_\mu N \to \mu^+ x\right)$$

allows a separation of $Q(x) = u(x) + d(x)$ from the antiquark distribution $\bar{Q}(x) = \bar{u}(x) + \bar{d}(x)$.

Neutral-Current Weak Interaction

For processes such as $\nu_\mu e^- \to \nu_\mu e^-$ and $\bar{\nu}_\mu e^+ \to \bar{\nu}_\mu e^+$, only neutral-current contributes. Similarly for reactions $\nu_\mu q \to \nu_\mu q$ and $\bar{\nu}_\mu q \to \bar{\nu}_\mu q$.



The invariant matrix element for $\nu_\mu q \to \nu_\mu q$ can be written as

$$M = \frac{G}{\sqrt{2}}\left[\bar{u}_\nu \gamma^\mu \left(1-\gamma^5\right) u_\nu\right]\left[\bar{u}_q \gamma_\mu \left(C_V^q - C_A^q \gamma^5\right) u_q\right] \tag{31}$$

M contains two terms:

$$\nu_L q_L \to \nu_L q_L \text{ and } \nu_L q_R \to \nu_L q_R$$

Note that Equation 31 shows that the neutral-current for neutrino is purely left-handed (since only left-handed neutrino is know to exist). For quarks, the neutral current can be a mixture of $V - A$ and $V + A$ currents:

$$\bar{u}_q \gamma_\mu \left(C_V^q - C_A^q \gamma^5\right) u_q = \underbrace{\bar{u}_q \gamma_\mu g_L^q \left(1-\gamma^5\right) u_q}_{\text{left-handed}} + \underbrace{\bar{u}_q \gamma_\mu g_R^q \left(1+\gamma^5\right) u_q}_{\text{right-handed}} \tag{32}$$

where

$$g_L^q = \frac{1}{2}\left(C_V^q + C_A^q\right) \text{ and } g_R^q = \frac{1}{2}\left(C_V^q - C_A^q\right) \tag{33}$$

represent the *V*- *A* and the *V* + *A* component of the neutral current, respectively.

Note that for $V - A$ coupling: q is left-handed, $\bar{q}$ is right-handed
for $V + A$ coupling: q is right-handed, $\bar{q}$ is left-handed

The expression for ν-induced neutral-current DIS on an isoscalar target is

$$\begin{aligned}\frac{d\sigma}{dxdy}\left(\nu N \to \nu x\right) = \frac{G^2 xs}{2\pi}\Big\{ & g_L^2\left(Q(x) + (1-y)^2 \bar{Q}(x)\right) \\ & + g_R^2\left(\bar{Q}(x) + (1-y)^2 Q(x)\right)\Big\}\end{aligned} \tag{34}$$

For $\bar{\nu} N \to \nu x$, one interchanges $s \leftrightarrow u$ (or $1 \leftrightarrow (1-y)^2$).

$$\begin{aligned}\frac{d\sigma^{NC}}{dxdy}\left(\bar{\nu} N \to \bar{\nu} x\right) = \frac{G^2 xs}{2\pi}\Big\{ & g_L^2\left(\bar{Q}(x) + (1-y)^2 Q(x)\right) \\ & + g_R^2\left(Q(x) + (1-y)^2 \bar{Q}(x)\right)\Big\}\end{aligned} \tag{35}$$

where

$$g_L^2 = \left(g_L^u\right)^2 + \left(g_L^d\right)^2 \qquad g_R^2 = \left(g_R^u\right)^2 + \left(g_R^d\right)^2 \tag{36}$$

In the electro-weak theory, the values of the vector coupling C_V^f and axial-coupling C_A^f are given as

$$C_V^f = T_f^3 - 2\sin^2\theta w Q_f \tag{37}$$

$$C_A^f = T_f^3 \tag{38}$$

where Q_f is the electric charge of the fermion, and T_f^3 is the third component of the weak isospin of the fermion. Leptons and quarks form weak-isospin doublets as follows:

$$\begin{pmatrix}\nu_e \\ e^-\end{pmatrix}\begin{pmatrix}\nu_\mu \\ \mu^-\end{pmatrix}\begin{pmatrix}\nu_\tau \\ \tau^-\end{pmatrix}\begin{pmatrix}u \\ d\end{pmatrix}\begin{pmatrix}c \\ s\end{pmatrix}\begin{pmatrix}t \\ b\end{pmatrix}$$

$T_f^3 = \frac{1}{2}$ for the upper members of the doublets and $T_f^3 = -\frac{1}{2}$ for the lower members.

Table 39

	Q_f	T_f^3	C_A^f	C_V^f
ν_e, ν_μ, ν_τ	0	+ ½	½	½
e^-, μ^-, τ^-	-1	$-1/2$	$-1/2$	$-1/2 + 2\sin^2\theta_w\ (\sim\ -0.03)$
u, c, t	$+2/3$	$+1/2$	$+1/2$	$1/2 - 4/3\sin^2\theta_w\ (\sim\ 0.19)$
d, s, b	$-1/3$	$-1/2$	$-1/2$	$-1/2 + 2/3\sin^2\theta_w\ (\sim\ -0.34)$

In table 39, the values for C_V^f were calculated using $\sin^2\theta_w = 0.231$, determined from experiments.

This table shows that for neutrinos, $C_A = C_V = ½$, and $g_L^\nu = 1/2,\ g_R^\nu = 0$, reflecting that ν is left-handed.
It also shows that for e^-, μ^-, τ^-, $C_V \simeq 0,\ C_A = -1/2$, hence the neutral current in this case is almost purely an axial-vector coupling, with $g_L^e \simeq -1/4,\ g_R^e \simeq +1/4$.

A comparison between Equations 30 and 35 shows that the neutral current cross-section reduces to the charged current cross-section when

$$g_L = 1,\ g_R = 0$$

We now revisit $\nu - e$ scattering taking into account the contribution of neutral current.

First, we consider the $\nu_\mu e^- \to \nu_\mu e^-$ and $\bar{\nu}_\mu e^- \to \bar{\nu}_\mu e^-$ reactions, which can only proceed via neutral current

$$m^{NC}\left(\nu_\mu e \to \nu_\mu e\right) = \frac{G_N}{\sqrt{2}}\left(\bar{u}_\nu \gamma^\mu \left(1-\gamma^5\right) u_\nu\right) \left(\bar{u}_e \gamma_\mu \left(C_V^e - C_A^e \gamma^5\right) u_e\right) \tag{40}$$

where $G_N = G_F = G$

Following procedures analogous to Equations 1 and 10, one obtains

$$\begin{aligned}\frac{d\sigma}{dy}\left(\nu_\mu e \to \nu_\mu e\right) &= \frac{G^2 s}{\pi}\left[g_L^2 + g_R^{e^2}\left(1-y\right)^2\right] \\ &= \frac{G^2 s}{4\pi}\left[\left(C_V^e + C_A^e\right)^2 + \left(C_V^e - C_A^e\right)^2 \left(1-y\right)^2\right]\end{aligned} \tag{41}$$

$$\frac{d\sigma}{dy}\left(\bar{\nu}_\mu e \to \bar{\nu}_\mu e\right) = \frac{G^2 s}{4\pi}\left[\left(C_V^e + C_A^e\right)^2 \left(1-y\right)^2 + \left(C_V^e - C_A^e\right)^2\right] \tag{42}$$

Note that Equation 42 is obtained from Equation 41 by $s \leftrightarrow u$ interchange (or $1 \leftrightarrow (1-y)^2$). These two cross-sections are also related by $C_A^e \leftrightarrow -C_A^e$ interchange.

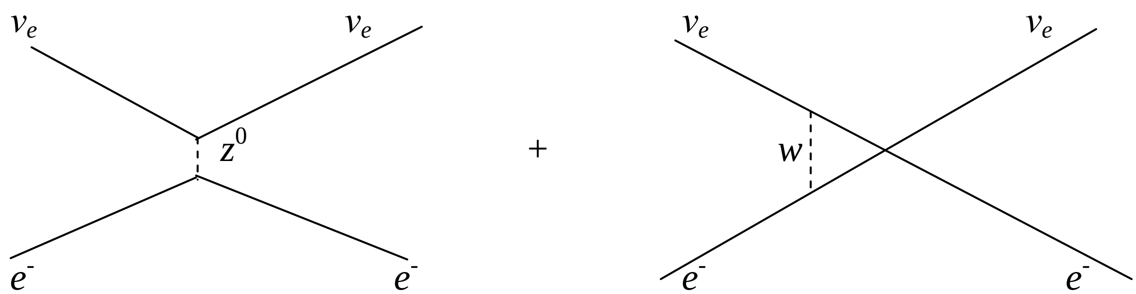
Integrating over y, Equations 41 and 42 become

$$\sigma\left(\nu_\mu e \to \nu_\mu e\right) = \frac{G^2 s}{3\pi}\left(C_v^{e^2} + C_v^e C_A^e + C_A^{e^2}\right) \tag{43}$$

$$\sigma\left(\bar{\nu}_\mu e \to \bar{\nu}_\mu e\right) = \frac{G^2 s}{3\pi}\left(C_v^{e^2} - C_v^e C_A^e + C_A^{e^2}\right) \tag{44}$$

Since $C_v^e \sim 0$ (Table 39), we expect that $\sigma\left(\nu_\mu e \to \nu_\mu e\right) \simeq \sigma\left(\bar{\nu}_\mu e \to \bar{\nu}_\mu e\right)$ (recall that $\sigma^{cc}\left(\nu_e e^- \to \nu_e e^-\right) = 3\sigma^{cc}\left(\bar{\nu}_e e^- \to \bar{\nu}_e e^-\right)$).

The $\nu_e e^- \to \nu_e e^-$ reaction contains contributions fro neutral current as well as charged current:



The corresponding amplitudes are

$$M^{NC}\left(\nu_e e \to \nu_e e\right) = \frac{G}{\sqrt{2}}\left(\bar{\nu}\gamma^\mu\left(1-\gamma^5\right)\nu\right)\left(\bar{e}\gamma_\mu\left(C_v^e - C_A^e\gamma^5\right)e\right) \tag{45}$$

$$\begin{aligned} M^{CC}\left(\nu_e e \to \nu_e e\right) &= -\frac{G}{\sqrt{2}}\left(\bar{e}\gamma^\mu\left(1-\gamma^5\right)\nu\right)\left(\bar{\nu}\gamma_\mu\left(1-\gamma^5\right)e\right) \\ &= \frac{G}{\sqrt{2}}\left(\bar{\nu}\gamma^\mu\left(1-\gamma^5\right)\nu\right)\left(\bar{e}\gamma_\mu\left(1-\gamma^5\right)e\right) \end{aligned} \tag{46}$$

The negative sign for the CC diagram is due to the interchange of the outgoing fermions.

The second line in Equation 46 is obtained using the Fierz transformation, which relates the ‘charge-exchange ordering’ to ‘charge-retention ordering’. (For a derivation of the Fierz theorems, see Giunti and Kim)

Adding the NC and the CC contributions, one has

$$M^{NC} + m^{CC} = \frac{G}{\sqrt{2}}\left(\bar{\nu}\gamma^\mu\left(1-\gamma^5\right)\nu\right)\left(\bar{e}\gamma_\mu\left[\left(C_v^e + 1\right) - \left(C_A^e + 1\right)\gamma^5\right]e\right) \tag{47}$$

Equation 47 has a form analogous to Equation 40, except that

$$C_v^e \to C_c^{e'} = C_v^e + 1 \qquad C_A^e \to C_A^{e'} = C_A^e + 1 \tag{48}$$

From Equations 43 and 48, we have

$$\frac{\sigma^{NC+CC}\left(\nu_e e \to \nu_e e\right)}{\sigma^{NC}\left(\nu_\mu e \to \nu_\mu e\right)} = \frac{\left(C_v^e + 1\right)^2 + \left(C_v^e + 1\right)\left(C_A^e + 1\right) + \left(C_A^e + 1\right)^2}{\left(C_v^e\right)^2 + C_v^e C_A^e + \left(C_A^e\right)^2} \tag{49}$$

Similarly for $\bar{\nu}_e e \to \bar{\nu}_e e$, we have

$$\frac{\sigma\left(\bar{\nu}_e e \to \bar{\nu}_e e\right)}{\sigma\left(\bar{\nu}_\mu e \to \bar{\nu}_\mu e\right)} = \frac{\left(C_v^e + 1\right)^2 - \left(C_v^e + 1\right)\left(C_A^e + 1\right) + \left(C_A^e + 1\right)^2}{\left(C_v^e\right)^2 - C_v^e C_A^e + \left(C_A^e\right)^2} \tag{50}$$

Summarizing:

Reaction	Current	Cross-section is proportional to
$\nu_\mu e^- \to \nu_\mu e^-$	NC	$\left(C_v^e\right)^2 + C_v^e C_A^e + \left(C_A^e\right)^2$
$\bar{\nu}_\mu e^- \to \bar{\nu}_\mu e^-$	NC	$\left(C_v^e\right)^2 - C_v^e C_A^e + \left(C_A^e\right)^2$
$\nu_e e^- \to \nu_e e^-$	$NC + CC$	$\left(C_v^e + 1\right)^2 + \left(C_v^e + 1\right)\left(C_A^e + 1\right) + \left(C_A^e + 1\right)^2$
$\bar{\nu}_e e^- \to \bar{\nu}_e e^-$	$NC + CC$	$\left(C_v^e + 1\right)^2 - \left(C_v^e + 1\right)\left(C_A^e + 1\right) + \left(C_A^e + 1\right)^2$

Using $C_A^e = -\frac{1}{2}$, $C_v^e = -0.03$ (from Table 39), Equation 49 gives

$$\frac{\sigma\left(\nu_e e \to \nu_e e\right)}{\sigma\left(\nu_\mu e \to \nu_\mu e\right)} = 6.3 \tag{51}$$

which is in good agreement with the experimental result:

$$\frac{\sigma\left(\nu_e e^- \to \nu_e e^-\right)}{\sigma\left(\nu_\mu e^- \to \nu_\mu e^-\right)} = \frac{0.93 \times 10^{-43}\,\mathrm{cm}^2\left(\mathrm{E}/10\ \mathrm{MeV}\right)}{0.16 \times 10^{-43}\,\mathrm{cm}^2\left(\mathrm{E}/10\ \mathrm{MeV}\right)} \tag{52}$$

The contribution of neutral current, together with the interference between the neutral current and the charged current terms, increases the $\nu_e e \to \nu_e e$ cross-section significantly over the $\nu_\mu e^- \to \nu_\mu e^-$ (and $\nu_\tau e^- \to \nu_\tau e^-$) cross-section. This fact has an interesting consequence for neutrino oscillation. When ν is propagating through a dense medium such as the sun and the earth's core, the effective 'index of refraction' for ν_e is different from that of ν_μ and ν_τ. This modifies the 'potential energy' of ν_e relative to ν_μ and ν_τ and effectively changes the missing angle between ν_e and ν_μ (ν_τ). One obtains

$$\tan^2\theta_M = \frac{\tan^2\theta_\nu}{1-\left(L_\nu / L_e\right)\sec^2\theta_\nu} \tag{53}$$

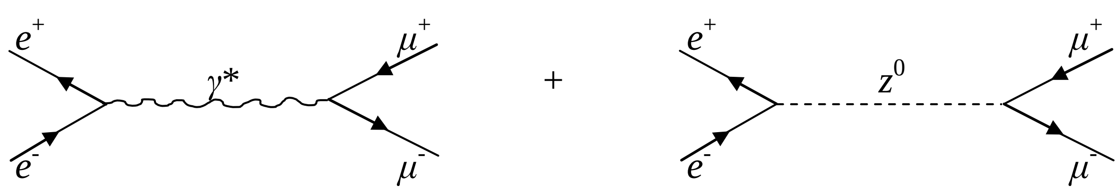
θ_ν is the mixing angle in vacuum

$$L_\nu = \frac{4\pi E_\nu}{\Delta m^2} \qquad L_e = \frac{\sqrt{2}\pi}{G n_e} \tag{54}$$

Δm^2 is the mass difference squared, ne is the electron density. Equation 54 shows that the mixing angle θ_ν can be significantly amplified in matter (when $L_\nu / L_e \sec^2\theta_\nu \to 1$). This is the MSW effect.

The existence of the neutral current also leads to some parity violating γ^* - z^0 interference effects. We consider two examples:

First consider the $e^+e^- \to \mu^+\mu^-$ reaction. The contributing diagrams are



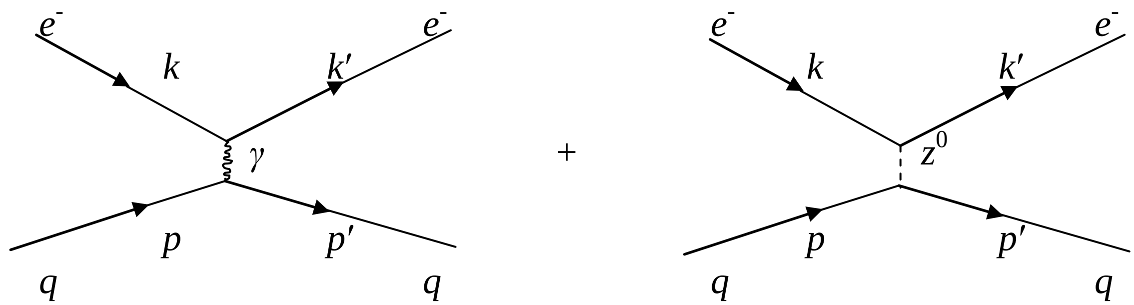
For a pure EM interaction, the angular distribution is $1 + \cos^2\theta$ c.m. This can be understood from the consideration that the helicities of the e^+, e^-, μ^+, μ^- have the following four possible combinations (and angular distributions).

$$e^+_R e^-_L \to \mu^+_R \mu^-_L \qquad (1+\cos\theta)^2$$
$$e^+_R e^-_L \to \mu^+_L \mu^-_R \qquad (1-\cos\theta)^2$$
$$e^+_L e^-_R \to \mu^+_L \mu^-_R \qquad (1+\cos\theta)^2$$

$$e_L^+ e_R^- \to \mu_R^+ \mu_L^- \qquad\qquad (1 - \cos\theta)^2$$

With a pure vector coupling, these four processes have equal probability, and the angular distribution is ~ $1 + \cos^2\theta$ (since the terms linear in $\cos\theta$ cancel).

When z^0 term is included, the four processes no longer have equal weighting. Hence the angular distribution is now given by ${d\sigma}/{d\Omega} \sim 1 + a\cos\theta + b\cos^2\theta$, with $a \neq 0$. This Forward-Backward asymmetry is observed experimentally, and leads to a determination of $\sin\theta_w$, the weak coupling angle. Another example is the parity violation, observed in $e^- N$ deep-inelastic scattering. The $eq \to eq$ scattering has two terms:



$$M = M_\gamma + M_{z^0}$$

$$M_\gamma = -Q_q e^2 \bar{u}(k')\gamma^\mu u(k)\frac{1}{q^2}\bar{u}(p')\gamma_\mu u(p)$$

$$M_{z^0} = -\frac{g^2}{4\cos^2\theta w}\left[\bar{u}(k')\gamma^\mu\left(C_v^e - C_A^e\gamma^5\right)u(k)\left(\frac{g_{\mu\nu} - q_\mu q_\nu / M_z^2}{q^2 - M_z^2}\right)\right.$$
$$\left.\bar{u}(p')\gamma^\nu\left(C_v^q - C_A^q\gamma^5\right)u(p)\right]$$

For $q^2 \ll M_z^2$, M_{z^0} becomes

$$M_{z^0} = -\frac{g^2}{4\cos^2\theta w M_z^2}\left[\bar{u}(k')\gamma^\mu\left(C_v^e - C_A^e\gamma^5\right)u(k)\bar{u}(p')\gamma_\mu\left(C_v^q - C_A^q\gamma^5\right)u(p)\right]$$

but
$$\frac{G}{\sqrt{2}} = \frac{g^2}{8M_z^2\cos^2\theta w}$$

Therefore,

$$
\begin{aligned}
M_{z^0} &= \sqrt{2}G\left[\bar{u}(k')\gamma^{\mu}\left(C_v^e - C_A^e\gamma^5\right)u(k)\right]\left[\bar{u}(p')\gamma_{\mu}\left(C_v^q - C_A^q\gamma^5\right)u(p)\right] \\
&= \sqrt{2}G\left[\bar{u}(k')\gamma^{\mu}C_L^e\frac{1}{2}\left(1-\gamma^5\right)u(k) + \bar{u}(k')\gamma^{\mu}C_R^e\frac{1}{2}\left(1+\gamma^5\right)u(k)\right] \\
&\qquad \left[\bar{u}(p')\gamma_{\mu}C_L^q\frac{1}{2}\left(1-\gamma^5\right)u(p) + \bar{u}(p')\gamma_{\mu}C_R^q\frac{1}{2}\left(1+\gamma^5\right)u(p)\right]
\end{aligned}
$$

where

$$
\begin{aligned}
C_L^e &= C_v^e + C_A^e \qquad & C_R^e &= C_v^e - C_A^e \\
C_L^q &= C_v^q + C_A^q \qquad & C_R^q &= C_v^q - C_A^q
\end{aligned}
$$

$$
\begin{aligned}
M_{z^0} = \sqrt{2}G\Big[&\bar{u}(k')\gamma^{\mu}C_L^e\frac{1}{2}\left(1-\gamma^5\right)u(k)\bar{u}(p')\gamma_{\mu}C_L^q\frac{1}{2}\left(1-\gamma^5\right)u(p) \\
&+ \bar{u}(k')\gamma^{\mu}C_L^e\frac{1}{2}\left(1-\gamma^5\right)u(k)\bar{u}(p')\gamma_{\mu}C_R^q\frac{1}{2}\left(1+\gamma^5\right)u(p) \\
&+ \bar{u}(k')\gamma^{\mu}C_R^e\frac{1}{2}\left(1+\gamma^5\right)u(k)\bar{u}(p')\gamma_{\mu}C_L^q\frac{1}{2}\left(1-\gamma^5\right)u(p) \\
&+ \bar{u}(k')\gamma^{\mu}C_R^e\frac{1}{2}\left(1+\gamma^5\right)u(k)\bar{u}(p')\gamma_{\mu}C_R^q\frac{1}{2}\left(1+\gamma^5\right)u(p)\Big]
\end{aligned}
$$

Since $$2\gamma^{\mu} = \gamma^{\mu}\left(1+\gamma^5\right) + \gamma^{\mu}\left(1-\gamma^5\right)$$

we have

$$M_\gamma = -\frac{Q_q e^2}{q^2}\left[\bar{u}(k')\gamma^\mu \frac{1}{2}(1-\gamma^5)u(k)\bar{u}(p')\gamma_\mu \frac{1}{2}(1-\gamma^5)u(p)\right.$$
$$+\bar{u}(k')\gamma^\mu \frac{1}{2}(1-\gamma^5)u(k)\bar{u}(p')\gamma_\mu \frac{1}{2}(1+\gamma^5)u(p)$$
$$+\bar{u}(k')\gamma^\mu \frac{1}{2}(1+\gamma^5)u(k)\bar{u}(p')\gamma_\mu \frac{1}{2}(1-\gamma^5)u(p)$$
$$\left.+\bar{u}(k')\gamma^\mu \frac{1}{2}(1+\gamma^5)u(k)\bar{u}(p')\gamma_\mu \frac{1}{2}(1+\gamma^5)u(p)\right]$$

where

$$r = -\frac{\sqrt{2}Gq^2}{e^2}$$

$$\overline{|M|^2} = \frac{1}{4}\sum_{spins}|M|^2$$

Only the ‘diagonal’ terms contribute to $|M|^2$, since the non-diagonal terms all contain factor of $(1 - \gamma^5)(1 + \gamma^5) = 0$. Therefore, only four terms remain when one evaluates $\overline{|M|^2}$.

$$|M|^2 = |M|^2_{LL\to LL} + |M|^2_{LR\to LR} + |M|^2_{RL\to RL} + |M|^2_{RR\to RR}$$

First consider $LL \to LL$

$$\left|M\right|^2_{LL\to LL} = \frac{1}{4}\frac{e^4}{q^4}\left(Q_q + r\ C_L^e C_L^q\right)^2$$

$$\sum_{spin}\left[\bar{u}(k')\gamma^\mu \frac{1}{2}\left(1-\gamma^5\right)u(k)\bar{u}(k)\gamma^\nu \frac{1}{2}\left(1-\gamma^5\right)u(k')\right]$$

$$\left[\bar{u}(p')\gamma_\mu \frac{1}{2}\left(1-\gamma^5\right)u(p)\bar{u}(p)\gamma_\nu \frac{1}{2}\left(1-\gamma^5\right)u(p')\right]$$

$$= \frac{1}{4}\frac{e^4}{q^4}\left(Q_q + rC_L^e C_L^q\right)^2 \frac{1}{16} Tr\left(\gamma^\mu\left(1-\gamma^5\right)\not{k}\gamma^\nu\left(1-\gamma^5\right)\not{k}'\right)$$

$$Tr\left(\gamma_\mu\left(1-\gamma^5\right)\not{p}\gamma_\nu\left(1-\gamma^5\right)\not{p}'\right)$$

$$= \frac{4e^4}{q^4}\left(Q_q + rC_L^e C_L^q\right)^2 (k\bullet p)(k'\bullet p')$$

but $$(k\bullet p)(k'\bullet p') = s^2/4$$

Therefore, $$\left(\frac{d\sigma}{d\Omega}\right)_{LL\to LL} = \frac{1}{64\pi^2 s}\frac{e^4}{q^4}\left(Q_q + r\ C_L^e C_L^q\right)^2 s^2$$

using $$q^4 = t^2 = \frac{s^2}{4}\left(1-\cos\theta\right)^2$$

$$1-y = \frac{1}{2}\left(1+\cos\theta\right) \qquad y = \frac{1}{2}\left(1-\cos\theta\right)$$

$$\frac{d\sigma}{dy} = \frac{d\sigma}{d\Omega}4\pi \qquad \alpha = \frac{e^2}{4\pi}$$

We have

$$\left(\frac{d\sigma}{dy}\right)_{LL\to LL} = \frac{\pi\alpha^2}{sy^2}\left(Q_q + r\ C_L^e C_L^q\right)^2 ...... \tag{55}$$

Similarly

$$\left(\frac{d\sigma}{dy}\right)_{RR\to RR} = \frac{\pi\alpha^2}{sy^2}\left(Q_q + r\ C_R^e C_R^q\right)^2 \ldots\ldots \tag{56}$$

Now, we consider $LR \rightarrow LR$

$$\begin{aligned}
|M|^2_{LR\to LR} &= \frac{1}{4}\frac{e^4}{q^4}\left(Q_q + r\ C_L^e C_R^q\right)^2 \times \\
&\quad \sum_{spin}\left[\bar{u}(k')\gamma^\mu \frac{1}{2}\left(1-\gamma^5\right)u(k)\bar{u}(k)\gamma^\nu \frac{1}{2}\left(1-\gamma^5\right)u(k')\right] \\
&\quad \left[\bar{u}(p')\gamma_\mu \frac{1}{2}\left(1+\gamma^5\right)u(p)\bar{u}(p)\gamma_\nu \frac{1}{2}\left(1+\gamma^5\right)u(p')\right] \\
&= 4\frac{e^4}{q^4}\left(Q_q + r\ C_L^e C_R^q\right)^2 (k\bullet p')(k'\bullet p)
\end{aligned}$$

but $$(k\bullet p')(k'\bullet p) = \frac{u^2}{4} = \frac{1}{4}s^2\left(\frac{1+\cos\theta}{2}\right)^2 = \frac{1}{4}s^2(1-y)^2$$

Therefore $$\left(\frac{d\sigma}{dy}\right)_{LR\to LR} = \frac{\pi\alpha^2}{sy^2}\left(Q_q + r\ C_L^e C_R^q\right)^2 (1-y)^2 \ldots\ldots \tag{57}$$

Similarly $$\left(\frac{d\sigma}{dy}\right)_{RL\to RL} = \frac{\pi\alpha^2}{sy^2}\left(Q_q + r\ C_R^e C_L^q\right)^2 (1-y)^2 \ldots\ldots \tag{58}$$

For an isoscalar target with $u = d$, and ignoring the antiquark contribution,

$$A = \frac{\left(\frac{d\sigma}{dy}\right)^u_{RR} + \left(\frac{d\sigma}{dy}\right)^d_{RR} + \left(\frac{d\sigma}{dy}\right)^u_{RL} + \left(\frac{d\sigma}{dy}\right)^d_{RL} - \left(\frac{d\sigma}{dy}\right)^u_{LR} - \left(\frac{d\sigma}{dy}\right)^d_{LR} - \left(\frac{d\sigma}{dy}\right)^u_{LL} - \left(\frac{d\sigma}{dy}\right)^d_{LL}}{\left(\frac{d\sigma}{dy}\right)^u_{RR} + \left(\frac{d\sigma}{dy}\right)^d_{RR} + \left(\frac{d\sigma}{dy}\right)^u_{RL} + \left(\frac{d\sigma}{dy}\right)^d_{RL} + \left(\frac{d\sigma}{dy}\right)^u_{LR} + \left(\frac{d\sigma}{dy}\right)^d_{LR} + \left(\frac{d\sigma}{dy}\right)^u_{LL} - \left(\frac{d\sigma}{dy}\right)^d_{LL}}$$

Using Equations (55), (56), (57), and (58), we obtain for the numerator of A (ignoring terms quadratic in r):

numerator of

$$A=\frac{\pi\alpha^2}{sy^2}\Bigg[\left(\frac{4}{9}+2r\left(\frac{2}{3}\right)C_R^eC_R^u+\frac{1}{9}-2r\left(\frac{1}{3}\right)C_R^eC_R^d\right)$$

$$+\left(\frac{4}{9}+2r\left(\frac{2}{3}\right)C_R^eC_L^u+\frac{1}{9}-2r\left(\frac{1}{3}\right)C_R^eC_L^d\right)\left(1-y^2\right)$$

$$-\left(\frac{4}{9}+2r\left(\frac{2}{3}\right)C_L^eC_L^u+\frac{1}{9}-2r\left(\frac{1}{3}\right)C_L^eC_L^d\right)$$

$$-\left(\frac{4}{9}+2r\left(\frac{2}{3}\right)C_L^eC_R^u+\frac{1}{9}-2r\left(\frac{1}{3}\right)C_L^eC_R^d\right)\left(1-y^2\right)\Bigg]$$

$$=\frac{\pi\alpha^2}{sy^2}\left(\frac{2r}{3}\right)\Big\{\left[2C_R^eC_R^u-C_R^eC_R^d-2C_L^eC_L^u+C_L^eC_L^d\right]$$

$$+\left(1-y^2\right)\left[2C_R^2C_L^u-C_R^eC_L^d-2C_L^eC_R^u+C_L^eC_R^d\right]\Big\}$$

$$=\frac{\pi\alpha^2}{sy^2}\left(\frac{2r}{3}\right)\Big\{\left[2C_v^e\left(-2C_A^u+C_A^d\right)+2C_A^e\left(-2C_v^uC_v^d\right)\right]$$

$$+\left(1-y^2\right)\left[2C_v^e\left(2C_A^u-C_A^d\right)+2C_A^e\left(-2C_v^u+C_v^d\right)\right]\Big\}$$

where we use $C_R = C_V - C_A$ $\quad C_L = C_V + C_A$

with the definition of

$$a_1 = C_A^e\left(2C_v^u - C_v^d\right)$$

$$a_2 = C_v^e\left(2C_A^u - C_A^d\right)$$

then, we have numerator of

$$A=\frac{\pi\alpha^2}{sy^2}\left(\frac{2r}{3}\right)\left[-2a_1-2a_2+\left(2a_2-2a_1\right)\left(1-y\right)^2\right] \tag{59}$$

The denominator of A is simply

denominator of $$A = \frac{\pi\alpha^2}{sy^2}\left(\frac{10}{9}\right)\left[1+(1-y)^2\right] \tag{60}$$

where we ignore terms linear or quadratic in r. This is justified since the numerator of A is linear in r and any term linear in r in the denominator only contribute to r^2 in A and can be ignored.

From Equations (5) and (6), we obtain (recall that $r = \frac{-\sqrt{2}Gq^2}{e^2}$)

$$A = \frac{6}{5}\left(\frac{\sqrt{2}Gq^2}{e^2}\right)\left(a_1 + a_2\frac{1-(1-y)^2}{1+(1-y)^2}\right)$$

Finally,

$$\begin{aligned} a_1 &= C_A^e\left(2C_v^u - C_v^d\right) \\ &= \left(-\frac{1}{2}\right)\left[(2)\left(\frac{1}{2}-\frac{4}{3}\sin^2\theta_w\right)-\left(-\frac{1}{2}+\frac{2}{3}\sin^2\theta_w\right)\right] \\ &= -\frac{3}{4}\left(1-\frac{20}{9}\sin^2\theta_w\right) \end{aligned}$$

Similarly,

$$\begin{aligned} a_2 &= C_v^e\left(2C_A^u - C_A^d\right) \\ &= \left(-\frac{1}{2}+2\sin^2\theta_w\right)\left(2\left(\frac{1}{2}\right)-\left(-\frac{1}{2}\right)\right) \\ &= -\frac{3}{4}\left(1-4\sin^2\theta_w\right) \end{aligned}$$

The muon decay, $\mu^- \to e^- + \overline{\nu}_e + \nu_\mu$, is an important example of purely lepton decay. The invariant matrix element is given as

$$m = \frac{G}{\sqrt{2}}\left[\overline{u}(k)\gamma^\mu\left(1-\gamma^5\right)u(p)\right]\left[\overline{u}(p')\gamma_\mu\left(1-\gamma^5\right)v(k')\right]$$

where the 4-vectors correspond to

$$\mu^-(p) \to e(p') + \bar{\nu}_e(k') + \nu_\mu(k)$$

It is straight forward to obtain

$$\overline{|m|^2} = 64G^2(k \bullet p')(k' \bullet p)$$

The decay width is given by

$$d\Gamma = \frac{1}{2E}\overline{|m|^2}dQ$$

where the Lorentz Invariant Phase Space *dQ* is

$$dQ = \frac{d^3p'}{(2\pi)^3 2E'} \; \frac{d^3k}{(2\pi)^3 2w} \; \frac{d^3k'}{(2\pi)^3 2w'} \; (2\pi)^4 \delta^4(p - p' - k - k')$$

Integrating over the delta function one obtains

$$\frac{d\Gamma}{dE'} = \frac{G^2}{12\pi^3} m^2 E'^2 \left(3 - \frac{4E'}{m}\right)$$

and the decay width

$$\Gamma = \frac{1}{\tau} = \int_0^{m/2} dE' \left(\frac{d\Gamma}{dE'}\right) = \frac{G^2 m^5}{192\pi^3}$$

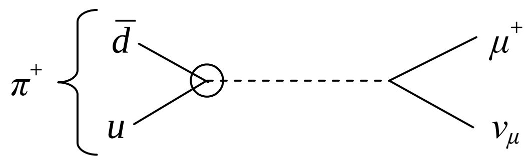
The decay width is proportional to the fifth power of the mass of the decaying particle.

Similar results can be obtained for other decays such as

$$\tau^- \to e^- \bar{\nu}_e \nu_\tau$$
$$b \to c \bar{\nu} \ell^-$$
$$t \to b e^+ \nu_e$$

We consider next the $\pi^+ \to \mu^+ + \nu_\mu$ decay, which is an example of semileptonic decay.

As in cases which involve hadrons, we need to parameterize the hadronic weak current based on general principles.



$\pi^+(q) \to \mu^+(p) + \nu_\mu(k)$

The leptonic current is

$$\bar{u}(p)\gamma_\mu\left(1-\gamma^5\right)v(k)$$

The hadronic current can only be a combination of V and A (in order to make the invariant amplitude a scalar or pseudoscalar).

$$M = \frac{G}{\sqrt{2}}\left(f\pi q^\mu\right)\left[\bar{u}(p)\gamma_\mu\left(1-\gamma^5\right)v(k)\right]$$

The only V or A which can be constructed from a single spin-0 object (like π) is q^μ, and $f\pi$ represents the pion structure factor.

$$\overline{|M|^2} = 4G^2 f_\pi^2 m_\mu^2 \left(p\bullet k\right)$$

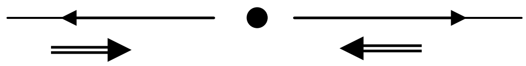
Note that the mass^2 of muon enters. For $\pi^+ \to e^+ + \nu_e$ decay, the m_e^2 term greatly reduces the decay probability.

$$P = \frac{1}{\tau} = \frac{G^2}{8\pi} f_\pi^2 m_\pi m_\mu^2 \left(1-\frac{m_\mu^2}{m_\pi^2}\right)^2$$

and

$$\frac{P\left(\pi^+ \to e^+\nu_e\right)}{P\left(\pi^+ \to \mu^+\nu_\mu\right)} = \left(\frac{m_e}{m_\mu}\right)^2 \left(\frac{m_\pi^2 - m_e^2}{m_\pi^2 - m_\mu^2}\right)^2 = 1.2\times 10^{-4}$$

e^+ π^+ ν_e



Since π^+ has spin-0, and ν_e has negative helicity, e^+ is required to be left-handed in order to conserve angular momentum. e^+ is therefore in the wrong helicity state, which inhibits the decay probability.

Similar suppression is observed for k decays:

$$\frac{\Gamma\left(k^- \to e^- \overline{\nu}_e\right)}{\Gamma\left(k^- \to \mu^- \overline{\nu}_\mu\right)} = 2.1\times 10^{-5}$$

One can also compare the $k^- \to \mu^- \overline{\nu}_\mu$ with $\pi^- \to \mu^- \overline{\nu}_\mu$:

$$R = \frac{\Gamma\left(k^- \to \mu^- \nu_\mu\right)}{\Gamma\left(\pi^- \to \mu^- \nu_\mu\right)} = \frac{\sin^2\theta_c}{\cos^2\theta_c}\frac{f_k^2 m_k}{f_\pi^2 m_\pi}\frac{\left[1-\left(m_\mu / m_k\right)^2\right]^2}{\left[1-\left(m_\mu / m_\pi\right)^2\right]^2}$$

From the experimental value of R, one determines

$$f_k / f_\pi = 1.28$$

Another decay related to $\pi \to \mu\nu$ is

$$\tau^- \to \pi^- \nu_\tau$$

or

$$\tau^- \to k^- \nu_\tau$$

The decay width is given as

$$\Gamma\left(\tau^- \to \pi^- \nu_\tau\right) = \frac{G^2 f_\pi^2 m_\tau^3}{16\pi}\left(1 - \frac{m_\pi^2}{m_\tau^2}\right)^2$$

Note that there is no helicity suppression in this decay, and

$$\Gamma\left(\tau^- \to \pi^- \nu_\tau\right) > \Gamma\left(\tau^- \to k^- \tau_\tau\right)$$

from phase-space consideration (and the cabbibo angle θ_c and f_k / f_π considerations).

Another example of semi-leptonic weak decay is

$$\pi^+ \to \pi^0 \ell^+ \nu$$

or $\qquad\qquad (M_a \to M_b \ell \nu)$

$$k^+ \to \pi^0 \ell^+ \nu$$

Now, we have two spin-0 hadrons. Therefore, we have two vectors, k_a and k_b, available for constructing the hadronic current:

$$\left\langle M_b \middle| J^\alpha \middle| M_a \right\rangle = N\left(f_a k_a^\alpha + f_b k_b^\alpha \right)$$

One can also extend this to baryonic semileptonic decay

$$B \to B' \ell \nu$$

We now have two spin-½ Dirac particles. One can form vector and axial-vector hadronic currents of various forms

vector:

$$\bar{u}(B')\gamma^\alpha u(B)$$
$$\bar{u}(B')\sigma^{\alpha\nu} q_\nu u(B)$$
$$\bar{u}(B') q^\alpha u(B)$$

axial-vector:

$$\bar{u}(B')\gamma^\alpha \gamma^5 u(B)$$
$$\bar{u}(B')\sigma^{\alpha\nu} q_\nu \gamma^5 u(B)$$
$$\bar{u}(B') q^\alpha \gamma^5 u(B)$$

$$\begin{aligned}\left\langle B' \middle| J^{\alpha} \middle| B \right\rangle = N\bar{u}\left(B'\right)\Big[& f_1\left(q^2\right)\gamma^{\alpha} + if_2\left(q^2\right)\sigma^{\alpha\nu}q_{\nu} + f_3\left(q^2\right)q^{\alpha} \\ & - g_1\left(q^2\right)\gamma^{\alpha}\gamma^5 - ig_2\left(q^2\right)\sigma^{\alpha\nu}q_{\nu}\gamma^5 \\ & - g_3\left(q^2\right)\gamma^5 q^{\alpha}\Big]u\left(B\right)\end{aligned}$$