Neutrino Oscillation in Matter

We discuss next the topics of "Neutrino Oscillation in Matter"

Consider $H = H_0 + W$, where

$$H_0 |\phi_1\rangle = m_1^0 |\phi_1\rangle, \quad H_0 |\phi_2\rangle = m_2^0 |\phi_2\rangle$$

$|\phi_1\rangle, |\phi_2\rangle$ are the eigenstates of H_0

W is represented by a Hermitian matrix in the basis of $|\phi_1\rangle, |\phi_2\rangle$

$$W = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} \qquad W_{12} = W_{21}^*$$

and H in the basis of $|\phi_1\rangle, |\phi_2\rangle$ becomes

$$H = \begin{pmatrix} m_1' & m_I^* \\ m_I & m_2' \end{pmatrix}$$

The eigenstates of H, $|\psi_1\rangle, |\psi_2\rangle$ can be expressed in terms of $|\phi_1\rangle, |\phi_2\rangle$ as

$$|\psi_1\rangle = \cos\theta |\phi_1\rangle + \sin\theta |\phi_2\rangle$$

$$|\psi_2\rangle = -\sin\theta |\phi_1\rangle + \cos\theta |\phi_2\rangle$$

H can be diagonalized as follows

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} m_1' & m_I^* \\ m_I & m_2' \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}$$

One can obtain easily

$$\tan 2\theta = \frac{2|m_I|}{m_2' - m_1'}$$

$$m_2 = \frac{1}{2}\{m_1' + m_2' + \sqrt{(m_2'-m_1')^2 + 4m_I^2}\,]$$

$$m_1 = \frac{1}{2}\{m_1' + m_2' - \sqrt{(m_2'-m_1')^2 + 4m_I^2}\,]$$

We note that

$$m_1 + m_2 = m_1' + m_2'$$

$$m_2 - m_1 = \sqrt{(m_2'-m_1')^2 + 4m_I^2} > (m_2' - m_1')$$

θ can reach $\pi/4$ if W_{11}, W_{22} are adjusted such that

$$m_2' - m_1' = m_2^0 + W_{22} - m_1^0 - W_{11} = 0$$

or $\dfrac{|W_{12}|}{m_2' - m_1'} \to \infty$

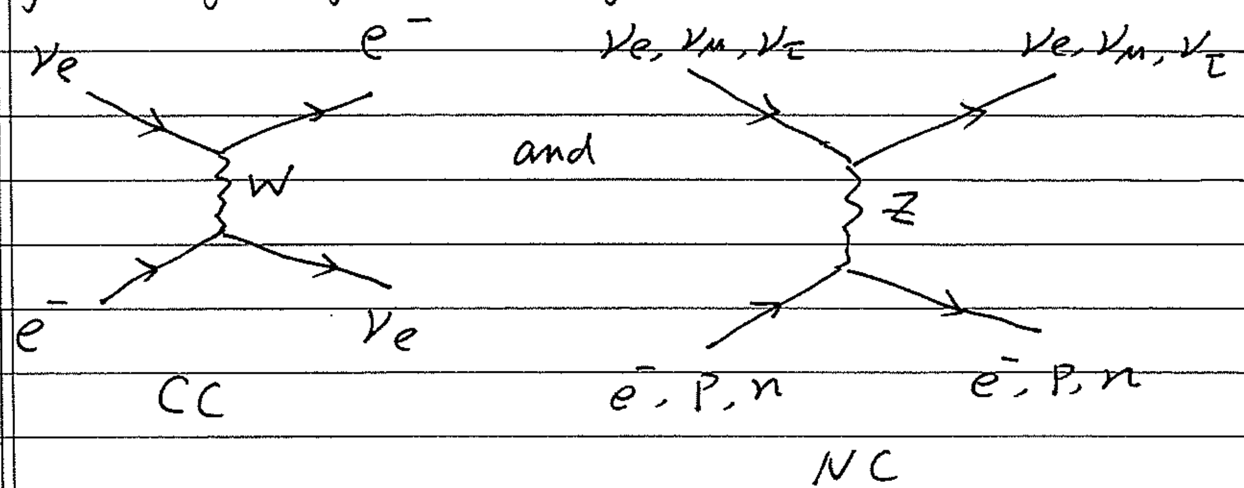
For neutrino in vaccum, then $W = 0$, $H = H_0$

$|\phi_1\rangle$, $|\phi_2\rangle$ are the mass eigenstates.

For neutrino in matter, then $W \neq 0$

and the angle θ is non-zero for $|W_{12}| > 0$

neutrinos can undergo coherent forward elastic scatterings via CC or NC via the following Feynman diagrams:



At low energies, the W/Z propagators become constant, and the 'point' interaction gives

$$\mathcal{L}^{CC} = -\frac{G_F}{\sqrt{2}} J_\mu^+ J^\mu$$

or, the effective CC Hamiltonian is

$$\mathcal{H}^{CC}(x) = \frac{G_F}{\sqrt{2}} [\bar{\nu}_e(x)\gamma^\mu(1-\gamma^5)e(x)][\bar{e}(x)\gamma_\mu(1-\gamma^5)\nu_e(x)]$$

It would be convenient to interchange $e(x)$ and $\nu_e(x)$, as indicated above.

This can be accomplished using the Fierz transformation

$$(\bar{\psi}_1 \Gamma^a \psi_2)(\bar{\psi}_3 \Gamma^a \psi_4) = -\frac{1}{16}\sum_b \mathrm{Tr}[(\Gamma^a\Gamma^b)^2](\bar{\psi}_1\Gamma^b\psi_4) \times (\bar{\psi}_3\Gamma^b\psi_2)$$

$$= \sum_b C_{ab}(\bar{\psi}_1\Gamma^b\psi_4)(\bar{\psi}_3\Gamma^b\psi_2)$$

From the table of C_{ab}, one can obtain

$$(\bar{\psi}_1 \gamma^\mu (1-\gamma^5)\psi_2)(\bar{\psi}_3 \gamma_\mu (1-\gamma^5)\psi_4)$$

$$= (\bar{\psi}_1 \gamma^\mu (1-\gamma^5)\psi_4)(\bar{\psi}_3 \gamma_\mu (1-\gamma^5)\psi_2)$$

using the Fierz transformation, we obtain

$$\mathcal{H}^{cc}(x) = \frac{G_F}{\sqrt{2}}[\bar{\nu}_e(x)\gamma^\mu(1-\gamma^5)\nu_e(x)][\bar{e}(x)\gamma_\mu(1-\gamma^5)e(x)]$$

For an unpolarized electron enviroment with an energy distribution $f(E_e)$, the averaged Hamiltonian becomes

$$\overline{\mathcal{H}^{cc}(x)} = \int d^3p_e\, f(E_e)\langle \mathcal{H}^{cc}(x)\rangle$$

where $\langle H^{cc}(x)\rangle$ is the expectation value of $\mathcal{H}^{cc}(x)$ summed over the helicity $h_e = \pm 1$:

$$\langle \mathcal{H}^{cc}(x)\rangle = \frac{G_F}{\sqrt{2}}[\overline{\nu_e(x)\gamma^\mu(1-\gamma^5)\nu_e(x)}]$$

$$\times \frac{1}{2}\sum_{h_e=\pm1}\langle e^-(p_e,h_e)|\bar{e}(x)\gamma_\mu(1-\gamma^5)e(x)|e^-(p_e,h_e)\rangle$$

Note that $e(x)$, $\nu_e(x)$, etc. are operators and single electron state has the normalization

$$|e^-(p_e,h_e)\rangle = \frac{1}{2E_eV}\, a_e^+(p_e,h_e)|0\rangle$$

$\int d^3p\, f(E_e) = N_eV$, N_e is the electron density, V is the volume, N_eV is the total number of electrons

From $e(x) = \int \frac{d^3 P}{(2\pi)^3 2E} \sum_{h_e=\pm 1} \left[a^{(h_e)}(p_e) u^{(h_e)}(p_e) e^{-i p_e \cdot x} + b^{(h_e)\dagger}(p_e) v^{(h_e)}(p_e) e^{-i p_e \cdot x} \right]$

one can obtain

$$\frac{1}{2} \sum_{h_e=\pm 1} \langle e^-(p_e, h_e) | \bar{e}(x) \gamma_\mu (1-\gamma^5) e(x) | e^-(p_e, h_e) \rangle$$

$$= \frac{1}{4 E_e V} \sum_{h_e=\pm 1} \overline{u_e^{(h_e)}}(p_e) \gamma_\mu (1-\gamma^5) u_e^{(h_e)}(p_e)$$

Note that $\overline{u_e^{(h_e)}}(p_e)$ and $u_e^{(h_e)}(p_e)$ have the same value of p_e, as required by coherent forward elastic scattering.

$$\sum_{h_e=\pm 1} \overline{u_e^{(h_e)}}(p_e) \gamma_\mu (1-\gamma^5) u_e^{(h_e)}(p_e)$$

$$= \mathrm{Tr}\left[\left(\sum_{h_e=\pm 1} u_e^{(h_e)}(p_e) \overline{u_e^{(h_e)}}(p_e) \right) \gamma_\mu (1-\gamma^5) \right]$$

$$= \mathrm{Tr}\left[(\not{p}_e + m_e) \gamma_\mu (1-\gamma^5) \right]$$

$$= \mathrm{Tr}\left[\not{p}_e \gamma_\mu + m_e \gamma_\mu - \not{p}_e \gamma_\mu \gamma^5 - m_e \gamma_\mu \gamma^5 \right]$$

$$= \mathrm{Tr}\left[\not{p}_e \gamma_\mu \right] = 4 p_{e,\mu}$$

(Note that $\mathrm{Tr}[\gamma_\mu] = 0$, $\mathrm{Tr}[\gamma_\mu \gamma^5] = 0$, $\mathrm{Tr}[\gamma_\nu \gamma_\mu \gamma^5] = 0$)

$$\overline{H^{cc}(x)} = \frac{G_F}{\sqrt{2}} \frac{1}{V} \int d^3 p_e\, f(E_e)\, \bar{\nu}_e(x) \frac{\not{p}_e}{E_e} (1-\gamma^5) \nu_e(x)$$

but $\int d^3 p_e\, f(E_e) \frac{\not{p}_e}{E_e} = \int d^3 p_e\, f(E_e) \left(\gamma^0 - \frac{\vec{p}_e \cdot \vec{\gamma}}{E_e} \right) = N_e V \gamma^0$

(since the integrand is odd under $\vec{p}_e \to -\vec{p}_e$)

Hence,

$$\overline{\mathcal{H}^{cc}(x)} = V_{cc}\ \overline{\nu_{eL}(x)}\gamma^0\ \nu_{eL}(x)$$

where $V_{cc} = \sqrt{2}\ G_F N_e$

Now, we consider the neutral current Hamiltonian,

$$\mathcal{H}^{NC}(x) = \frac{G_F}{\sqrt{2}} \sum_{\alpha=e,\mu,\tau} [\bar{\nu}_\alpha(x)\gamma^\mu(1-\gamma^5)\nu_\alpha(x)] \sum_f \{\bar{f}(x)\gamma_\mu(g_V^f - g_A^f\gamma^5) f(x)\},$$

f denotes the fermion, with vector and axial vector coupling to Z as g_V^f and g_A^f

$$g_V^f = I_3^f - 2 q_f \sin^2\theta_W$$

$$g_A^f = I_3^f$$

f	g_V	g_A
ν	$\frac{1}{2}$	$\frac{1}{2}$
l^-	$-\frac{1}{2} + 2\sin^2\theta_W$	$-\frac{1}{2}$
u	$\frac{1}{2} - \frac{4}{3}\sin^2\theta_W$	$\frac{1}{2}$
d	$-\frac{1}{2} + \frac{2}{3}\sin^2\theta_W$	$-\frac{1}{2}$
$p(uud)$	$\frac{1}{2} - 2\sin^2\theta_W$	$\frac{1}{2}$
$n(udd)$	$-\frac{1}{2}$	$-\frac{1}{2}$
$H(p+e^-)$	0	0

$$g_{V,A}(p) = -g_{V,A}(e^-)$$

$$g_{V,A}(n) = -g_{V,A}(\nu)$$

From the calculation of $\overline{H^{cc}(x) = V_{cc}\, \bar{\nu}_{eL}(x)\gamma^0 \nu_{eL}(x)}$,
it is clear that the $g_A^f \gamma^5$ term in

$$\sum_f \{ \bar{f}(x) \gamma_\mu (g_V^f - g_A^f \gamma^5) f(x)]$$ can not contribute.

Therefore, we only need to consider the g_V^f term.

* Unlike the CC Hamiltonian, which is only relevant for ν_e and $f = e$, the NC Hamiltonian applies to ν_e, ν_μ, ν_τ and $f = e, p, n$.

* Since NC does not distinguish ν_e, ν_μ, ν_τ, we only need to label f.

$$V_{CC} = \sqrt{2}\, G_F N_e \longrightarrow V_{NC}^f = \sqrt{2}\, G_F g_V^f N_f$$

(note that for V_{CC}, $g_V^f = 1$ for $V-A$ interaction)

* Since the net g_V for $(e^- + p)$ is zero, we only need to consider the contribution from neutrons (which is a small fraction in the SUN, largely consisting of $e^- + p$ plasma)

* Therefore

$$\sum_f V_{NC}^f = -\frac{1}{2}\sqrt{2}\, G_F N_n$$

is the neutral-current potential for any neutrino flavor α.

Summarizing, the total potentials from NC and CC for ν_e, ν_μ, ν_τ are

$$V(\nu_e) = \sqrt{2}\, G_F \left(N_e - \tfrac{1}{2} N_n\right)$$
$$V(\nu_\mu) = \sqrt{2}\, G_F \left(-\tfrac{1}{2} N_n\right)$$
$$V(\nu_\tau) = \sqrt{2}\, G_F \left(-\tfrac{1}{2} N_n\right)$$

What would be the potential energy of a ν_α (with h helicity) propagating through a medium? It would be the expectation value given as

$$V_\alpha^{(h)} = \langle \nu_\alpha(p,h) | \int d^3x\, \bar{H}(x) | \nu_\alpha(p,h)\rangle$$

where $\bar{H}(x) = \bar{H}^{CC}(x) + \bar{H}^{NC}(x)$, h is the helicity of ν with flavor α..

$$V_\alpha^{(h)} = \frac{V_\alpha}{4E}\, \overline{u^{(h)}_{\nu_\alpha}}(p)\, \gamma^0 (1-\gamma^5)\, u^{(h)}_{\nu_\alpha}(p)$$

$$\overline{u^{(h)}_{\nu_\alpha}}(p)\, \gamma^0 (1-\gamma^5)\, u^{(h)}_{\nu_\alpha}(p)$$
$$= \mathrm{Tr}\left[u^{(h)}_{\nu_\alpha}(p)\, \overline{u^{(h)}_{\nu_\alpha}}(p)\, \gamma^0 (1-\gamma^5)\right]$$
$$= \mathrm{Tr}\left[(\not{p} + m_{\nu_\alpha}) \left(\frac{1+\gamma^5 \not{s}_h}{2}\right) \gamma^0 (1-\gamma^5)\right]$$

(note that $\frac{1+\gamma^5 \not{s}_h}{2}$ projects out helicity h)

$$= \mathrm{Tr}\left[\tfrac{1}{2}\gamma^0 \not{p} - \tfrac{1}{2} m_{\nu_\alpha} \gamma^5 \not{s}_h \gamma^0 \gamma^5\right]$$

$$\left(s_h = h\left(\frac{|\vec{p}|}{m}, \frac{E}{m}\frac{\vec{p}}{|\vec{p}|}\right)\right)$$

$$= 4p^0 + \mathrm{Tr}\left(-\tfrac{1}{2} m_{\nu_\alpha} \not{s}_h \gamma^0\right]$$
$$= 4p^0 - 2h|\vec{p}|$$
$$= \begin{cases} 4E & \text{if } h=-1 \\ m_{\nu_e}^2/E & \text{if } h=+1 \end{cases}$$

Hence

$$V_\alpha^{(-)} \simeq V_\alpha \quad (\text{for } h=-1)$$

$$V_\alpha^{(+)} \simeq V_\alpha \frac{m_{\nu_\alpha}^2}{4E^2} \quad (\text{for } h=+1)$$

One can also find out the potential energy of a $\overline{\nu_\alpha}$ (with helicity h) propagating through the medium:

$$\overline{V}_\alpha^{(h)} = \langle \overline{\nu}_\alpha (p,h) | \int d^3x \; \overline{H}(x) | \overline{\nu}_\alpha (p,h) \rangle$$

$$= \frac{-V_\alpha}{4E} \overline{v}_{\nu_\alpha}^{(h)}(p) \gamma^0 (1-\gamma^5) v_{\nu_\alpha}^{(h)}(p)$$

$$= \begin{cases} -V_\alpha & (\text{for } h=+1) \\ -V_\alpha \frac{m_{\nu_\alpha}^2}{4E^2} & (\text{for } h=-1) \end{cases}$$

Therefore, the potential energy for a right-handed antineutrino has the same magnitude, but with an opposite sign with respect to the potential energy of a left-handed neutrino.

We are ready to consider neutrino oscillation in matter.

Consider a left-handed neutrino of flavor α produced at $t=0$:

$$|\nu_\alpha\rangle = \sum_k U^*_{\alpha k} |\nu_k\rangle$$

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_I$$

$\mathcal{H}_0$ is the Hamiltonian in vacuum, and $\mathcal{H}$ is the total Hamiltonian in matter

The mass eigenstates $|\nu_k\rangle$ correspond to the case in vacuum:

$$\mathcal{H}_0 |\nu_k\rangle = E_k |\nu_k\rangle$$

while the flavor states $|\nu_\alpha\rangle$ are eigenstates of $\mathcal{H}_I$:

$$\mathcal{H}_I |\nu_\alpha\rangle = V_\alpha |\nu_\alpha\rangle$$

It is to be expected that neither $|\nu_k\rangle$ nor $|\nu_\alpha\rangle$ are the eigenstates of $\mathcal{H}$.

The time evolution for $|\nu_\alpha(t)\rangle$ is

$$i \frac{d}{dt} |\nu_\alpha(t)\rangle = \mathcal{H} |\nu_\alpha(t)\rangle$$

where $|\nu_\alpha(0)\rangle = |\nu_\alpha\rangle$

define the neutrino transition amplitude $\psi_{\alpha\beta}(t)$ as $\psi_{\alpha\beta}(t) = \langle \nu_\beta | \nu_\alpha(t)\rangle$

Then, the transition probability is

$$P_{\nu_\alpha \to \nu_\beta}(t) = |\psi_{\alpha\beta}(t)|^2$$

From $i\frac{d}{dt}|\nu_\alpha(t)\rangle = H|\nu_\alpha(t)\rangle$

we have

$$\langle \nu_\beta | i\frac{d}{dt} | \nu_\alpha(t)\rangle = \langle \nu_\beta | H | \nu_\alpha(t)\rangle$$

$$= \langle \nu_\beta | H_0 | \nu_\alpha(t)\rangle + \langle \nu_\beta | H_I | \nu_\alpha(t)\rangle$$

The second term on the RHS is

$$\langle \nu_\beta | H_I | \nu_\alpha(t)\rangle = \sum_\eta \langle \nu_\beta | H_I | \nu_\eta\rangle \langle \nu_\eta | \nu_\alpha(t)\rangle$$

$$= \sum_\eta V_\beta \delta_{\beta\eta} \psi_{\alpha\eta}(t) \qquad (\eta \text{ is the flavor index})$$

The first term on the RHS is

$$\langle \nu_\beta | H_0 | \nu_\alpha(t)\rangle = \sum_k \langle \nu_\beta | H_0 | \nu_k\rangle \langle \nu_k | \nu_\alpha(t)\rangle$$

$$= \sum_{k,\eta} \langle \nu_\beta | H_0 | \nu_k\rangle \langle \nu_k | \nu_\eta\rangle \langle \nu_\eta | \nu_\alpha(t)\rangle$$

$$= \sum_{k,\eta} U_{\beta k} E_k U_{\eta k}^* \psi_{\alpha\eta}(t) \qquad (\eta \text{ is the mass eigenstate index (in vacuum)})$$

Therefore, we obtain

$$i\frac{d}{dt}\psi_{\alpha\beta}(t) = \sum_\eta \left(\sum_k U_{\beta k} E_k U_{\eta k}^* + \delta_{\beta\eta} V_\beta \right) \psi_{\alpha\eta}(t)$$

Using $E_k \simeq E + \frac{m_k^2}{2E}$, $t \simeq x$, $p \simeq E$,

$$V_\alpha = V_{CC}\,\delta_{\alpha e} + V_{NC}$$

we obtain

$$i \frac{d}{dx} \psi_{\alpha\beta}(x) = \left(p + \frac{m_1^2}{2E} + V_{NC} \right) \psi_{\alpha\beta}(x)$$
$$+ \sum_{\eta} \left(\sum_{k} U_{\beta k} \frac{\Delta m_{k1}^2}{2E} U_{\eta k}^{*} + \delta_{\beta e} \delta_{\eta e} V_{CC} \right) \psi_{\alpha\eta}(x)$$

The first term on the RHS is not relevant, since it corresponds to a phase in $\psi_{\alpha\beta}(x)$.
It can be eliminated by shifting the phase of $\psi_{\alpha\beta}(x)$, i.e., a shift of

$$\exp\left\{ -i \left(p + m_1^2/2E \right) x - i \int_0^x V_{NC}(x')\, dx' \right]$$

Therefore, the evolution of the transition amplitude is

$$i \frac{d}{dx} \psi_{\alpha\beta}(x) = \sum_{\eta} \left(\sum_{k} U_{\beta k} \frac{\Delta m_{k1}^2}{2E} U_{\eta k}^{*} + \delta_{\beta e} \delta_{\eta e} V_{CC} \right) \psi_{\alpha\eta}(x)$$

For a given α, the above equation contains $\beta = e, \mu, \tau$ parts
This is equivalent to a matrix equation

$$i \frac{d}{dx} \psi_{\alpha} = H_F \psi_{\alpha}$$

where $H_F = \frac{1}{2E} \left(U M^2 U^{+} + A \right)$

$$\psi_{\alpha} = \begin{pmatrix} \psi_{\alpha e} \\ \psi_{\alpha\mu} \\ \psi_{\alpha\tau} \end{pmatrix}; \quad M^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \Delta m_{21}^2 & 0 \\ 0 & 0 & \Delta m_{31}^2 \end{pmatrix}; \quad A = \begin{pmatrix} A_{CC} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A_{CC} = 2E V_{CC} = 2\sqrt{2} E G_F N_e$$

We consider the solution for $i \frac{d}{dx} \psi_\alpha = \mathcal{H}_F \psi_\alpha$
for the two-neutrino mixing case:

$$\nu_e - \nu_\mu \quad (\nu_1 \text{ and } \nu_2)$$

$$\psi_\alpha = \begin{pmatrix} \psi_{\alpha e} \\ \psi_{\alpha\mu} \end{pmatrix} ; \quad M^2 = \begin{pmatrix} 0 & 0 \\ 0 & \Delta m^2 \end{pmatrix} ; \quad A = \begin{pmatrix} A_{CC} & 0 \\ 0 & 0 \end{pmatrix}$$

$$A_{CC} = 2E V_{CC} = 2\sqrt{2}\, E\, G_F N_e$$

$$\mathcal{H}_F = \frac{1}{2E} \left(U M^2 U^\dagger + A \right)$$

One can make M^2 and A more symmetric by adding diagonal matrices

$$M^2 = \begin{pmatrix} 0 & 0 \\ 0 & \Delta m^2 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 0 \\ 0 & \Delta m^2 \end{pmatrix} + \begin{pmatrix} -\frac{\Delta m^2}{2} & 0 \\ 0 & -\frac{\Delta m^2}{2} \end{pmatrix}$$

$$= \begin{pmatrix} -\Delta m^2/2 & 0 \\ 0 & \Delta m^2/2 \end{pmatrix} = (M^2)'$$

$$A = \begin{pmatrix} A_{CC} & 0 \\ 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} A_{CC} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -\frac{A_{CC}}{2} & 0 \\ 0 & -\frac{A_{CC}}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{A_{CC}}{2} & 0 \\ 0 & -\frac{A_{CC}}{2} \end{pmatrix} = A'$$

$$\mathcal{H}_F \longrightarrow \mathcal{H}_F + \left(-\frac{\Delta m^2}{2} - \frac{A_{CC}}{2} \right) I = \mathcal{H}_F'$$

Therefore $\mathcal{H}_F'$ and $\mathcal{H}_F$ differ by a term proportional to a unit matrix aI, where $a = \left(+ \frac{\Delta m^2}{2} + \frac{A_{CC}}{2} \right) / 2E$. a can depend on x since A_{CC} can depend on x (due to $N_e(x)$)

Let $\psi_\alpha' = \exp\{i\int_0^x a(x')dx'] \psi_\alpha$

then

$$i\frac{d}{dx}\psi_\alpha' = \exp[i\int_0^x a(x')dx'] \, i\frac{d}{dx}\psi_\alpha$$
$$- a(x)\exp\{i\int_0^x a(x')dx'] \psi_\alpha$$
$$= \exp\{i\int_0^x a(x')dx'] H_F \psi_\alpha$$
$$- a(x)\exp\{i\int_0^x a(x')dx'] \psi_\alpha$$
$$= [H_F - a(x) I] \psi_\alpha' = H_F' \psi_\alpha'$$

Hence ψ_α' satisfies the same differential equation as ψ_α, provided that $H_F \to H_F' = H_F - a(x) I$

Since ψ_α' and ψ_α only differ by a phase factor, the transition probability, $|\psi_\alpha|^2$, remains the same for $|\psi_\alpha'|^2$.

Therefore, we are free to change M^2 to $\begin{pmatrix} -\frac{\Delta m^2}{2} & 0 \\ 0 & \frac{\Delta m^2}{2} \end{pmatrix}$ and A to $\begin{pmatrix} \frac{A_{cc}}{2} & 0 \\ 0 & -\frac{A_{cc}}{2} \end{pmatrix}$. note the $U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$.

$$UM^2U^+ + A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} -\frac{\Delta m^2}{2} & 0 \\ 0 & \frac{\Delta m^2}{2} \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} + \begin{pmatrix} \frac{A_{cc}}{2} & 0 \\ 0 & -\frac{A_{cc}}{2} \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} -\Delta m^2\cos 2\theta + A_{cc} & \Delta m^2 \sin 2\theta \\ \Delta m^2 \sin 2\theta & \Delta m^2 \cos 2\theta - A_{cc} \end{pmatrix}$$

The differential equation becomes

$$i \frac{d}{dx}\begin{pmatrix} \psi_{ee} \\ \psi_{e\mu} \end{pmatrix} = \frac{1}{4E}\begin{pmatrix} -\Delta m^2 \cos 2\theta + A_{cc} & \Delta m^2 \sin 2\theta \\ \Delta m^2 \sin 2\theta & \Delta m^2 \cos 2\theta - A_{cc} \end{pmatrix}\begin{pmatrix} \psi_{ee} \\ \psi_{e\mu} \end{pmatrix}$$

note the θ is the mixing angle in vacuum.

$$\nu_e = \cos\theta\, \nu_1 + \sin\theta\, \nu_2 \; ; \quad \nu_\mu = -\sin\theta\, \nu_1 + \cos\theta\, \nu_2$$

$\Delta m^2 = \Delta m_{21}^2 = m_2^2 - m_1^2$, which is mass-squared difference in vacuum.

For $\mathcal{H}_F = \frac{1}{4E}\begin{pmatrix} -\Delta m^2 \cos 2\theta + A_{cc} & \Delta m^2 \sin 2\theta \\ \Delta m^2 \sin 2\theta & \Delta m^2 \cos 2\theta - A_{cc} \end{pmatrix}$

~~we would like to diagonalize it~~

$$U_M^T \mathcal{H}_F U_M = \mathcal{H}_M, \text{ where } \mathcal{H}_M \text{ is diagonal}$$

write $U_M = \begin{pmatrix} \cos\theta_M & \sin\theta_M \\ -\sin\theta_M & \cos\theta_M \end{pmatrix}$

and $\mathcal{H}_M = \frac{1}{4E}\begin{pmatrix} -\Delta m_M^2 & 0 \\ 0 & \Delta m_M^2 \end{pmatrix}$

It is straighforward to show that

$$\Delta m_M^2 = \left\{ (\Delta m^2 \cos 2\theta - A_{cc})^2 + (\Delta m^2 \sin 2\theta)^2 \right]^{1/2}$$

$$A_{cc} = 2\sqrt{2}\, E\, G_F\, N_e(x)$$

~~The minimal value of Δm_M^2 is $\Delta m^2 \sin 2\theta$, which occurs at $\Delta m^2 \cos 2\theta = A_{cc}$~~

It is simple to show

$$\tan 2\theta_M = \frac{\Delta m^2 \sin 2\theta}{\Delta m^2 \cos 2\theta - A_{CC}} = \frac{\tan 2\theta}{1 - \frac{A_{CC}}{\Delta m^2 \cos 2\theta}}$$

When $\Delta m^2 \cos 2\theta - A_{CC} = 0$, $\theta_M = \pi/4$, maximal mixing

Therefore, even with a tiny value of θ, one can obtain maximal mixing $\theta_M = \pi/4$, if the electron density is such that $A_{CC}^R = \Delta m^2 \cos 2\theta$ (we call this 'Resonance', hence the notation R)

Since $A_{CC} = 2\sqrt{2}\, E\, G_F\, N_e$, the 'resonance' electron density occurs at

$$N_e^R = \frac{\Delta m^2 \cos 2\theta}{2\sqrt{2}\, E\, G_F}$$

* This density can be reached if Δm^2 is small enough, or if E is large enough. For example, for $\Delta m^2 = 10^{-5}\,\text{eV}^2$, $\theta \simeq 0.1^\circ$, $E = 1\,\text{MeV}$.

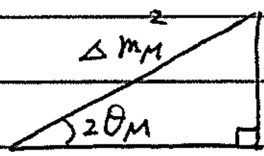
We obtain $N_e^R = 65.6\ N_A/\text{cm}^3$

where N_A is the Avogadro number.

This density is reachable in SUN.

* Note that if $\theta = \pi/4$, then the resonant electron density is zero, and maximal mixing occurs in vacuum.

One can also show graphically :



$\Delta m^2 \sin 2\theta \; (= \Delta m_M^2 \sin 2\theta_M)$

$\Delta m^2 \cos 2\theta - A_{cc}$
$(= \Delta m_M^2 \cos 2\theta_M)$

(From $\tan 2\theta_M = \dfrac{\Delta m^2 \sin 2\theta}{\Delta m^2 \cos 2\theta - A_{cc}}$, one obtains

$$\sin 2\theta_M = \frac{\Delta m^2 \sin 2\theta}{\left[(\Delta m^2 \cos 2\theta - A_{cc})^2 + (\Delta m^2 \sin 2\theta)^2\right]^{1/2}} = \frac{\Delta m^2 \sin 2\theta}{\Delta m_M^2}$$

and

$$\cos 2\theta_M = \frac{\Delta m^2 \cos 2\theta - A_{cc}}{\left[(\Delta m^2 \cos 2\theta - A_{cc})^2 + (\Delta m^2 \sin 2\theta)^2\right]^{1/2}} = \frac{\Delta m^2 \cos 2\theta - A_{cc}}{\Delta m_M^2}$$

Note that $\Delta m^2 \sin 2\theta$ part of this triangle is x-independent (or density independent), while the $\Delta m^2 \cos 2\theta - A_{cc}$ part of the triangle depends on the local electron density N_e.

One can see that the $\mathcal{H}_F$ matrix can be written as

$$\mathcal{H}_F = \frac{1}{4E}\begin{pmatrix} -\Delta m^2 \cos 2\theta + A_{cc}, & \Delta m^2 \sin 2\theta \\ \Delta m^2 \sin 2\theta & \Delta m^2 \cos 2\theta - A_{cc} \end{pmatrix} = \frac{1}{4E}\begin{pmatrix} -\Delta m_M^2 \cos 2\theta_M & \Delta m_M^2 \sin 2\theta_M \\ \Delta m_M^2 \sin 2\theta_M & \Delta m_M^2 \cos 2\theta_M \end{pmatrix}$$

After diagonalizing $\mathcal{H}_F$, how would the evolution equation, $i \frac{d}{dx} \psi_\alpha = \mathcal{H}_F \psi_\alpha$, be modified?

We have

$$i \frac{d}{dx} \begin{pmatrix} \psi_{ee} \\ \psi_{e\mu} \end{pmatrix} = U_M H_M U_M^+ \begin{pmatrix} \psi_{ee} \\ \psi_{e\mu} \end{pmatrix}$$

we can define

$$\phi_e = \begin{pmatrix} \phi_{e1} \\ \phi_{e2} \end{pmatrix} = U_M^+ \begin{pmatrix} \psi_{ee} \\ \psi_{e\mu} \end{pmatrix} = U_M^+ \psi_e \; ; \; \psi_e = \begin{pmatrix} \psi_{ee} \\ \psi_{e\mu} \end{pmatrix}$$

ϕ_{e1}, ϕ_{e2} correspond to the transition amplitude for $|\nu_e\rangle$ at $t=0$ to transform into $|\nu_1\rangle_M$, $|\nu_2\rangle_M$, which are the mass eigenstates in matter (for a given N_e electron density) later at time t.

The evolution equation can be expressed in terms of ϕ_e as (since $\psi_e = U_M \phi_e$)

$$i \frac{d}{dx} \psi_e = U_M \mathcal{H}_M \phi_e$$

or

$$U_M^+ i \frac{d}{dx} (U_M \phi_e) = \mathcal{H}_M \phi_e \qquad \left(\mathcal{H}_M = \frac{1}{4E} \begin{pmatrix} -\Delta m_M^2 & 0 \\ 0 & \Delta m_M^2 \end{pmatrix} \right)$$

The LHS becomes

$$U_M^+ i \frac{d}{dx} (U_M) \begin{pmatrix} \phi_{e1} \\ \phi_{e2} \end{pmatrix} + i U_M^+ U_M \frac{d}{dx} \begin{pmatrix} \phi_{e1} \\ \phi_{e2} \end{pmatrix}$$

$$= i \begin{pmatrix} \cos\theta_M & -\sin\theta_M \\ \sin\theta_M & \cos\theta_M \end{pmatrix} \frac{d}{dx} \begin{pmatrix} \cos\theta_M & \sin\theta_M \\ -\sin\theta_M & \cos\theta_M \end{pmatrix} \begin{pmatrix} \phi_{e1} \\ \phi_{e2} \end{pmatrix} + i \frac{d}{dx} \begin{pmatrix} \phi_{e1} \\ \phi_{e2} \end{pmatrix}$$

$$i\, U_M^{+} \frac{d}{dx} [U_M]$$

$$= i \begin{pmatrix} \cos\theta_M & -\sin\theta_M \\ \sin\theta_M & \cos\theta_M \end{pmatrix} \begin{pmatrix} -\sin\theta_M\, \theta_M' & \cos\theta_M\, \theta_M' \\ -\cos\theta_M\, \theta_M' & -\sin\theta_M\, \theta_M' \end{pmatrix}$$

$$= i\, \theta_M' \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix} \quad \text{where } \theta_M' = \frac{d\theta_M}{dx}$$

Hence the evolution equation becomes

$$i \frac{d}{dx} \begin{pmatrix} \phi_{e1} \\ \phi_{e2} \end{pmatrix} = \frac{1}{4E} \begin{pmatrix} -\Delta m_M^2 & -4Ei\,\theta_M' \\ 4Ei\,\theta_M' & \Delta m_M^2 \end{pmatrix} \begin{pmatrix} \phi_{e1} \\ \phi_{e2} \end{pmatrix}$$

$$\frac{d\theta_M}{dx} = \frac{1}{2} \frac{\sin 2\theta_M}{\Delta m_M^2} \frac{dA_{cc}}{dx}$$

1) The simplest case is a constant N_e, i.e. $\theta_M' = 0$, corresponding to propagation in vacuum or uniform medium:

ϕ_{e1} and ϕ_{e2} are decoupled, Δm_M^2 is constant, the evolution for ϕ_{e1}, ϕ_{e2} just lead to a change in phase $\phi_{e_1}(x) = \phi_{e_1}(0) \exp\left(\frac{i \Delta m_M^2}{4E} x\right)$

$$\begin{pmatrix} \phi_{e1}(0) \\ \phi_{e2}(0) \end{pmatrix} = \begin{pmatrix} \cos\theta_M^{(i)} & -\sin\theta_M^{(i)} \\ \sin\theta_M^{(i)} & \cos\theta_M^{(i)} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta_M^{(i)} \\ \sin\theta_M^{(i)} \end{pmatrix}$$

$$P_{\nu_e \to \nu_\mu}(x) = \sin^2 2\theta_M \sin^2\left(\frac{\Delta m_M^2\, x}{4E}\right)$$

2) Another case, so-called adiabatic evolution, is for a non-zero θ_M', but θ_M' is negligible compared to the diagonal terms:

$$\gamma = \frac{\Delta m_M^2}{4E\,|\theta_M'|} \gg 1$$

where γ is the "adiabaticity" parameter

Note that ϕ_{e1} and ϕ_{e2} are decoupled in this case, however, Δm_M^2 depends on x, hence the solutions for $\phi_{e1}(x)$ and $\phi_{e2}(x)$ are

$$\phi_{e1}(x) = \exp\left[i \int_0^x \frac{\Delta m_M^2(x')}{4E}\, dx' \right] \phi_{e1}(0)$$

$$\phi_{e2}(x) = \exp\left[-i \int_0^x \frac{\Delta m_M^2(x')}{4E}\, dx' \right] \phi_{e2}(0)$$

Let $\int_0^x \frac{\Delta m_M^2(x')}{4E}\, dx' = A$

then, the survival amplitude $\phi_{ee}(x)$ is

$$\phi_{ee}(x) = \cos\theta_M^{(f)}\, \phi_{e1}(x) + \sin\theta_M^{(f)}\, \phi_{e2}(x)$$

$$= \cos\theta_M^{(f)} \cos\theta_M^{(i)} \exp(iA) + \sin\theta_M^{(f)} \sin\theta_M^{(i)} \exp(-iA)$$

and the survival probability is

$$|\phi_{ee}(x)|^2 = P_{\nu_e\to\nu_e}(x) = \frac{1}{2} + \frac{1}{2}\cos 2\theta_M^{(i)} \cos 2\theta_M^{(f)} + \frac{1}{2}\sin 2\theta_M^{(i)} \sin 2\theta_M^{(f)} \times \cos(2A)$$

If $\theta_M^{(i)} = \theta_M^{(f)} = \theta$ (vacuum)

then

$$P_{\nu_e \to \nu_e} = \frac{1}{2} + \frac{1}{2}\cos^2 2\theta + \frac{1}{2}\sin^2 2\theta \cos(2A)$$

$$= \frac{1}{2} + \frac{1}{2}(1 - \sin^2 2\theta) + \frac{1}{2}\sin^2 2\theta \cos(2A)$$

$$= 1 - \frac{1}{2}\sin^2 2\theta (1 - \cos 2A)$$

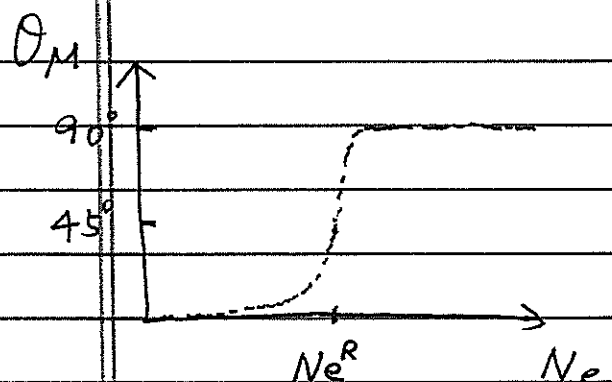
$$= 1 - \frac{1}{2}\sin^2 2\theta (2\sin^2 A) = 1 - \sin^2 2\theta \sin^2\left(\frac{\Delta m^2 L}{4E}\right)$$

as expected.

In general, the 'adiabatic' transition implied an averaged survival probability

$$\overline{P}^{\,adiabatic}_{\nu_e \to \nu_e} = \frac{1}{2} + \frac{1}{2}\cos 2\theta_M^{(i)} \cos 2\theta_M^{(f)}$$

Consider a ν_e produced near the center of sun, where $N_e > N_e^R$ (and small θ)

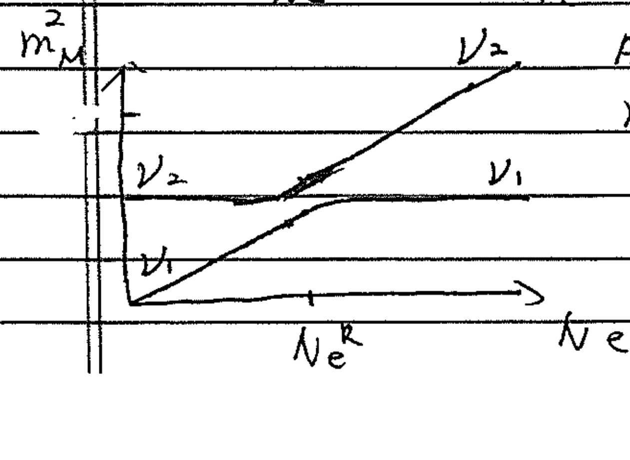


$\theta_M^{(I)} \cong 90°$

$$\tan 2\theta_M = \frac{\Delta m^2 \sin 2\theta}{\Delta m^2 \cos 2\theta - A_{CC}}$$

$\nu_e = \cos\theta_M \nu_1 + \sin\theta_M \nu_2$

$\theta_M^{(I)} \simeq 90° \Rightarrow \nu_e \sim \nu_2$ at $t=0$



Adiabaticity implies that no transition (coupling) between ν_1 and ν_2

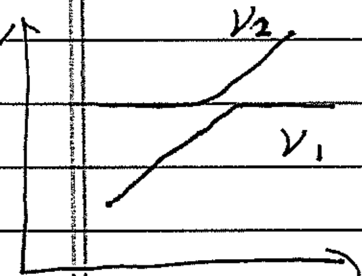
$\theta_M^{(f)} = \theta \to 0 \Rightarrow \nu_2^{(f)} \sim \nu_\mu$

$P_{\nu_e \to \nu_e} \simeq \frac{1}{2} + \frac{1}{2}\cos(180°)\cos(0°) = 0$

$P_{\nu_e \to \nu_\mu} \simeq 1$

3) In the non-adiabatic case, when γ near the resonance is small, then one can either find out the solution numerically (solve the coupled differential equation), or use an approximation as follows:

Since γ at the region away from the resonance is large (adiabatic), we only need to consider the situation when ν_e crosses the resonance.



The probability for $\nu_2 \to \nu_1$ at the resonance-crossing is

$$P_c = P_{\nu_2 \to \nu_1} = P_{\nu_1 \to \nu_2}$$

The probability for $\nu_2 \to \nu_2$ whiling crossing the resonance is

$$1 - P_c = P_{\nu_2 \to \nu_2} = P_{\nu_1 \to \nu_1}$$

$$P_c = \frac{\exp\left[-\frac{\pi}{2}\gamma_R F\right] - \exp\left[-\frac{\pi}{2}\gamma_R \frac{F}{\sin^2\theta}\right]}{1 - \exp\left[-\frac{\pi}{2}\gamma_R \frac{F}{\sin^2\theta}\right]}$$

F depends on the shape of $N_e(x)$, and $F \simeq 1$

For $\gamma_R \to 0$, we have $P_c \longrightarrow \cos^2\theta$

and $\overline{P_{\nu_e \to \nu_e}}(\gamma_R \to 0) \simeq 1 - \frac{1}{2}\sin^2 2\theta$

just like in vaccum. For abrupt density change at the resonance, ν_e maintains it's flavor, and oscillate like ν_e afterwards.

$$\overline{P}_{\nu_e \to \nu_e} = \frac{1}{2} + \left(\frac{1}{2} - P_c\right) \cos 2\theta_M^{(i)} \cos 2\theta$$

1) In vacuum: $\theta_M^{(i)} = \theta$, $P_c = 0$

$$\overline{P}_{\nu_e \to \nu_e} = \frac{1}{2} + \frac{1}{2}\cos^2 2\theta = 1 - \frac{1}{2}\sin^2 2\theta$$

2) Adiabatic: $P_c = 0$

$$\overline{P}_{\nu_e \to \nu_e} = \frac{1}{2} + \frac{1}{2}\cos 2\theta_M^{(i)} \cos 2\theta$$

a) $Ne^{(i)} \gg Ne^{R}$, $\theta_M^{(i)} = \pi/2$

$$\overline{P}_{\nu_e \to \nu_e} = \frac{1}{2} - \frac{1}{2}\cos 2\theta$$

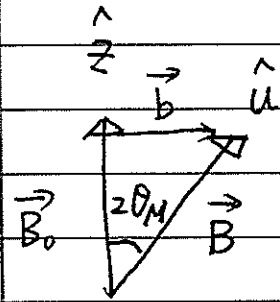
b) $Ne^{(i)} \ll Ne^{R}$, $\theta_M^{(i)} = \theta$

$$\overline{P}_{\nu_e \to \nu_e} = \frac{1}{2} + \frac{1}{2}\cos^2 2\theta = \frac{1}{2} + \frac{1}{2}(1 - \sin^2 2\theta)$$
$$= 1 - \frac{1}{2}\sin^2 2\theta$$

3) Abrupt: $P_c = \cos^2\theta$

$$\overline{P}_{\nu_e \to \nu_e} = 1 - \frac{1}{2}\sin^2 2\theta$$

In QM, there is a well known analogy between mixing of a two-level system and the behavior of spin-½ particle in magnetic field.



$$|\nu_e\rangle \longleftrightarrow \uparrow = |+\rangle$$

$$|\nu_\mu\rangle \longleftrightarrow \downarrow = |-\rangle$$

$$|\nu_1\rangle \longleftrightarrow |+\rangle_u$$

$$|\nu_2\rangle \longleftrightarrow |-\rangle_u$$

At $t=0$, $|\nu_e\rangle$ is like a spin-½ particle pointing along the $\hat{z}$ axis with the probability of $|+\rangle$ being $\cos^2(\frac{\theta}{2}) = 1$ and $|-\rangle$ being $\sin^2(\frac{\theta}{2}) = 0$

$$H = -\gamma \vec{B}\cdot\vec{S} = -\frac{\gamma\hbar}{2}\begin{pmatrix} B_z & B_x - iB_y \\ B_x + iB_y & -B_z \end{pmatrix}$$

Let $B_z = |\vec{B}_0|$, $B_y = 0$, $B_x = |\vec{b}|$

then

$$H \sim \begin{pmatrix} B_0 & B_x \\ B_x & -B_0 \end{pmatrix}$$

which is the same form as

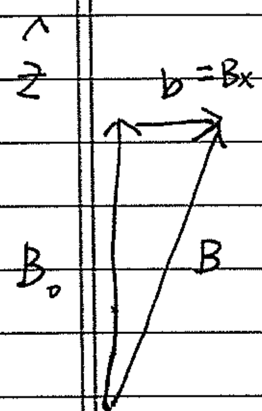
$$\frac{1}{4E}\begin{pmatrix} -\Delta m^2 \cos 2\theta + A_{cc} & \Delta m^2 \sin 2\theta \\ \Delta m^2 \sin 2\theta & \Delta m^2 \cos 2\theta - A_{cc} \end{pmatrix}$$

$|\nu_e\rangle \longleftrightarrow \uparrow = |+\rangle$

$|\nu_\mu\rangle \longleftrightarrow \downarrow = |-\rangle$

$H = -\gamma \vec{B}\cdot\vec{S}$, hence $\uparrow$ and $\downarrow$ are eigenstates of H when $\vec{S}$ is along the $\hat{z}$ axis

$$H_0 = -\gamma \vec{B_0}\cdot\vec{S} = \begin{pmatrix} -\frac{\gamma\hbar}{2}B_0 & 0 \\ 0 & +\frac{\gamma\hbar}{2}B_0 \end{pmatrix}$$



eigenstate for

$$H = H_0 + H_I = -\gamma(\vec{B_0}+\vec{b})\cdot\vec{S}$$

$$H = \frac{+\gamma\hbar}{2}\begin{pmatrix} -B_0 & b_x - ib_y \\ b_x + ib_y & +B_0 \end{pmatrix}$$

$$\vec{b} = B_x\,\hat{x}$$

$$H = \frac{+\gamma\hbar}{2}\begin{pmatrix} -B_0 & B_x \\ B_x & +B_0 \end{pmatrix}$$

eigenval

$$H_{diag} = U^T H U$$

$$U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$H_{diag} = \frac{+\gamma\hbar}{2}\begin{pmatrix} -B & 0 \\ 0 & B \end{pmatrix} \qquad B = (B_0^2 + B_x^2)^{1/2}$$

$$\tan 2\theta = \frac{B_x}{B_0}$$

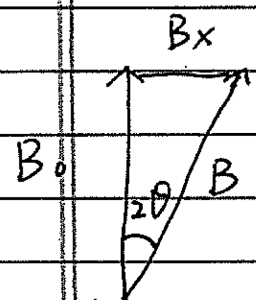
$$B_x = B_0 \tan 2\theta \,, \qquad B = B_0 (1+\tan^2 2\theta)^{1/2}$$
$$= B_0 / \cos 2\theta$$

or $\quad B_0 = B \cos 2\theta$

$\quad\;\; B_x = B \sin 2\theta$

Hence

$$H = \frac{\gamma \hbar}{2} \begin{pmatrix} -B\cos 2\theta & B \sin 2\theta \\ B \sin 2\theta & B \cos 2\theta \end{pmatrix}$$



For a spin-$\frac{1}{2}$ particle

The eigenstate becomes

$|+\rangle_u$ and $|-\rangle_u$ where

$\hat{u}$ points along the direction of $\hat{B}$

$$|+\rangle_u = \cos\theta\, |+\rangle + \sin\theta\, |-\rangle$$

$$|-\rangle_u = -\sin\theta\, |+\rangle + \cos\theta\, |-\rangle$$

(if $\theta = 0$, $|+\rangle_u = |+\rangle$

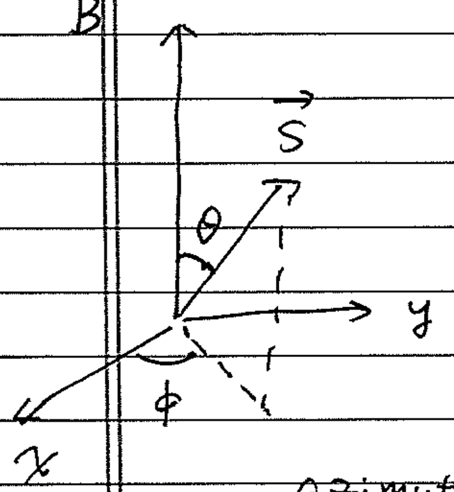
if $\theta = \pi/2$, $|+\rangle_u = |-\rangle$

if $\theta = \pi/4$, $|+\rangle_u = \frac{1}{\sqrt{2}} (|+\rangle + |-\rangle)$

$$|\langle + | + \rangle_u|^2 = \cos^2\theta \;; \quad |\langle - | + \rangle_u|^2 = \sin^2\theta$$

precession Larmor frequency $\omega = \gamma B$

Precession of spin-½ in B field



$$H = -\gamma \vec{B} \cdot \vec{S}$$

$$\hat{S} = \begin{pmatrix} \cos\frac{\theta}{2}\, e^{-i\phi/2} \\ \sin\frac{\theta}{2}\, e^{i\phi/2} \end{pmatrix} ; \quad \vec{S} = \frac{\hbar}{2}\hat{S}$$

θ, ϕ are the polar and azimuthal angles

$$\psi(t=0) = \begin{pmatrix} \cos\frac{\theta}{2}\, e^{-i\phi/2} \\ \sin\frac{\theta}{2}\, e^{i\phi/2} \end{pmatrix}$$

$$\psi(t) = e^{-\frac{iHt}{\hbar}} \begin{pmatrix} \cos\frac{\theta}{2}\, e^{-i\frac{\phi}{2}} \\ \sin\frac{\theta}{2}\, e^{i\frac{\phi}{2}} \end{pmatrix}$$

$$e^{-\frac{iHt}{\hbar}} = \exp\left\{-\frac{it}{\hbar}\begin{pmatrix} -\gamma B\frac{\hbar}{2} & 0 \\ 0 & \gamma B\frac{\hbar}{2} \end{pmatrix}\right]$$

$$= \exp\left\{\begin{matrix} i\frac{\gamma B}{2}t & 0 \\ 0 & -i\frac{\gamma B}{2}t \end{matrix}\right] = \begin{pmatrix} \exp i\left(\frac{\gamma B}{2}t\right) & 0 \\ 0 & \exp -i\left(\frac{\gamma B}{2}t\right) \end{pmatrix}$$

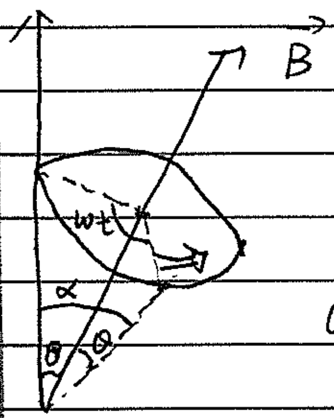
$$\psi(t) = \begin{pmatrix} \cos\frac{\theta}{2}\, e^{-i(\phi - \gamma B t)} \\ \sin\frac{\theta}{2}\, e^{-i(\phi - \gamma B t)} \end{pmatrix}$$

precession with $w = \gamma B$

After adding $\vec{b}$ to $\vec{B}_r$, the diagonal H' becomes

$$\frac{1}{4E}\begin{pmatrix} -\Delta m_M^2 & 0 \\ 0 & \Delta m_M^2 \end{pmatrix}$$

and the rotating angle is θ_M, and the new axis points along $\hat{u}$, making an angle of $2\theta_M$ with respect to $\hat{z}$.



precession frequency ω is $u|\vec{B}|$ or $\frac{\Delta m_M^2}{4E}$

$$\cos\alpha = \cos^2\theta + \sin^2\theta \cos\omega t$$

and the probability to find a $|\nu_e\rangle$ at time t is $(\theta = 2\theta_M)$

$$P_{\nu_e \to \nu_e} = \cos^2\frac{\alpha}{2} = \frac{1+\cos\alpha}{2}$$

$$= \frac{1+\cos^2\theta + \sin^2\theta\cos\omega t}{2} = \frac{1+1-\sin^2(2\theta_M) + \sin^2 2\theta_M \cos\omega t}{2}$$

$$= 1 - \sin^2(2\theta_M)\cdot \sin^2\left(\frac{\Delta m_M^2 t}{4E}\right)$$

$|\nu_e\rangle = |+\rangle, \quad |\nu_\mu\rangle = |-\rangle$

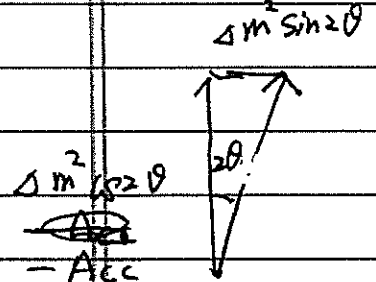
$$H = \frac{1}{4E}\begin{pmatrix} -\Delta m_M^2 \cos 2\theta_M & \Delta m_M^2 \sin 2\theta_M \\ \Delta m_M^2 \sin 2\theta_M & \Delta m_M^2 \cos 2\theta_M \end{pmatrix}$$

$$= \frac{1}{4E}\begin{pmatrix} -\Delta m^2 \cos 2\theta + A_{cc} & \Delta m^2 \sin 2\theta \\ \Delta m^2 \sin 2\theta & \Delta m^2 \cos 2\theta - A_{cc} \end{pmatrix}$$

Vaccum mass eigenstate ~~become~~

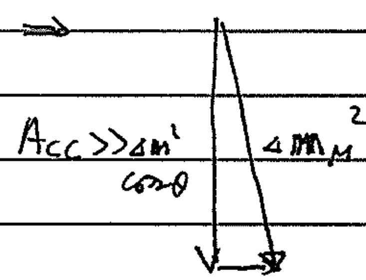
If $\theta_M = 0 = \theta$

mass eigenstate = flavor eigenstate



$A_{cc} = 0$

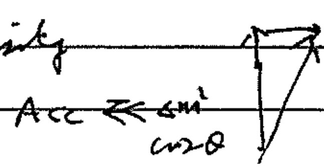
In matter 1) high density $\Rightarrow$



2) resonance $\Rightarrow$

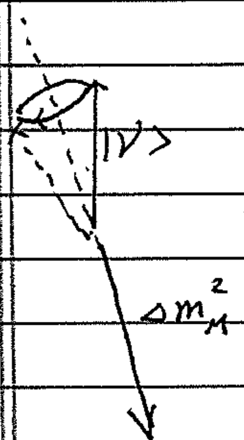
$A_{cc} = \Delta m^2 \cos 2\theta \longrightarrow \Delta m_M^2$

3) low density



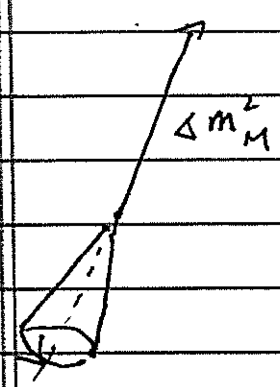
Now, consider ν_e produced at a density which is well above the resonance density, i.e.

$$N_e \gg N_e^R \quad , \quad 2\theta_M \sim \pi$$



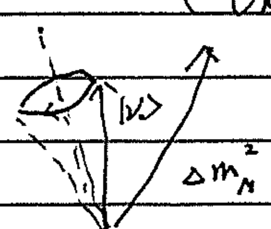
$|\nu\rangle_t$ is roughly $|\nu_2\rangle$, and precess in a small cone along ~~the axis of the mass eigenstate~~

For adiabatic crossing of the resonance, ~~$|\nu\rangle_t$ precess along an axis whose direction~~ continues to change. Eventually, after crossing the resonance, it becomes



therefore, $|\nu\rangle$ becomes largely $|\nu_\mu\rangle$ for adiabatic transition

If the density changes too rapidly ($\theta_M' \gg \frac{4E}{\Delta m_M^2}$) then we have



Since $|\nu\rangle$ can not follow the rapid change of mass eigenstates. Hence $|\nu\rangle$ remains as $|\nu_e\rangle$ after crossing the resonance.