Neutrino Mixing and Oscillation

We will discuss next the topics of "neutrino oscillation"

For neutrino flavor eigenstates $|\nu_\alpha\rangle$, $(\alpha = e, \mu, \tau)$, they can be expressed in terms of the mass eigenstates as

$$|\nu_\alpha\rangle = \sum_k U^*_{\alpha k} |\nu_k\rangle$$

k is summed over 1 to N, $N = 3 + N_s$

U is unitary and $\langle \nu_k | \nu_j \rangle = \delta_{kj}$,

$\langle \nu_\alpha | \nu_\beta \rangle = \delta_{\alpha\beta}$

U has the dimension $3 \times N$, $UU^\dagger = I$, $U^\dagger U = I$

At $t=0$, $|\nu_\alpha\rangle$ is produced in a flavor eigenstate

$$|\nu_\alpha\rangle = \sum_k U^*_{\alpha k} |\nu_k\rangle$$

$$|\nu_\alpha(t)\rangle = \sum_k U^*_{\alpha k} e^{-iE_k t} |\nu_k\rangle$$

since $U^\dagger U = 1$, $|\nu_k\rangle = \sum_\alpha U_{\alpha k} |\nu_\alpha\rangle$

$$|\nu_\alpha(t)\rangle = \sum_\beta \left(\sum_k U^*_{\alpha k} e^{-iE_k t} U_{\beta k} \right) |\nu_\beta\rangle$$

$$P_{\nu_\alpha \to \nu_\beta}(t) = |\langle \nu_\beta | \nu_\alpha(t) \rangle|^2$$

$$= \sum_{k,j} U^*_{\alpha k} U_{\beta k} U_{\alpha j} U^*_{\beta j} e^{-i(E_k - E_j)t}$$

$$E_k \sim p_k + \frac{m_k^2}{2p_k} = p + \frac{m_k^2}{2p} \qquad (p \text{ is the } \nu \text{ momentum})$$

$$E_k - E_j \simeq \frac{\Delta m_{kj}^2}{2E} \qquad p \simeq E$$

$t = L$ for relativistic limit,

$$P_{\nu_\alpha \to \nu_\beta}(L, E) = \sum_{k,j} U_{\alpha k}^* U_{\beta k} U_{\alpha j} U_{\beta j}^* \exp\left(-i \frac{\Delta m_{kj}^2 L}{2E}\right)$$

The oscillation probability depends on the product

$$U_{\alpha k}^* U_{\beta k} U_{\alpha j} U_{\beta j}^*$$

* For Majorana neutrinos

$$U_{\alpha k}^M = U_{\alpha k}^D e^{i\lambda_k}$$

or

$$U^M = U^D \begin{pmatrix} 1 & & \\ & e^{i\lambda_1} & \\ & & e^{i\lambda_2} \end{pmatrix}$$

where U^D is the PMNS mixing matrix for Dirac neutrino with 3-mixing angles and one Dirac CP-phase (for $N = 3$ case)

a) It is clear that

$$U_{\alpha k}^{M*} U_{\beta k}^M U_{\alpha j}^M U_{\beta j}^{M*} = U_{\alpha k}^D U_{\beta k}^D U_{\alpha j}^D U_{\beta j}^D$$

~~Hence, the oscillation probability is the same for Dirac and Majorana neutrinos~~

b) As a consequence of a), neutrino oscillation can not be used to determine the Majorana CP-phases λ_1 and λ_2, even if neutrino is found to be Majorana-type.

* One can also write the oscillation probability as

$$P_{\nu_\alpha \to \nu_\beta}(L, E) = \sum_k |U_{\alpha k}|^2 |U_{\beta k}|^2 + 2 \operatorname{Re} \sum_{k>j} U_{\alpha k}^* U_{\beta k} U_{\alpha j} U_{\beta j}^* \exp\left(-2\pi i \frac{L}{L_{kj}^{osc}}\right)$$

L_{kj}^{osc} is the "oscillation length" at which the phase generated by Δm_{kj}^2 becomes 2π.

$$L_{kj}^{osc} = \frac{4\pi E}{\Delta m_{kj}^2}$$

The non-oscillatory term in $P_{\nu_\alpha \to \nu_\beta}$, which would survive over an average over E or L, is

$$\langle P_{\nu_\alpha \to \nu_\beta} \rangle = \sum_k |U_{\alpha k}|^2 |U_{\beta k}|^2$$

* The unitary of U also implies

$$\sum_\beta P_{\nu_\alpha \to \nu_\beta}(L, E) = 1$$

and

$$\sum_\alpha P_{\nu_\alpha \to \nu_\beta}(L, E) = 1$$

Using $\sum_k U_{\alpha k} U_{\beta k}^* = \delta_{\alpha\beta}$, and squaring it, one can write

$$P_{\nu_\alpha \to \nu_\beta}(L, E) = \delta_{\alpha\beta} - 4 \sum_{k>j} \mathrm{Re}\left[U_{\alpha k}^* U_{\beta k} U_{\alpha j} U_{\beta j}^*\right] \sin^2\left(\frac{\Delta m_{kj}^2 L}{4E}\right)$$
$$+ 2 \sum_{k>j} \mathrm{Im}\left[U_{\alpha k}^* U_{\beta k} U_{\alpha j} U_{\beta j}^*\right] \sin\left(\frac{\Delta m_{kj}^2 L}{2E}\right)$$

If $\alpha = \beta$ (survival probability), then

$$U_{\alpha k}^* U_{\beta k} U_{\alpha j} U_{\beta j}^* = |U_{\alpha k}|^2 |U_{\alpha j}|^2$$

is real, hence only the real term remains

$$P_{\nu_\alpha \to \nu_\alpha}(L, E) = 1 - 4 \sum_{k>j} |U_{\alpha k}|^2 |U_{\alpha j}|^2 \sin^2\left(\frac{\Delta m_{kj}^2 L}{4E}\right)$$

If we have anti-neutrino $\bar{\nu}_\alpha$ at $t=0$, how does it oscillate?

$$|\bar{\nu}_\alpha\rangle = \sum_k U_{\alpha k} |\bar{\nu}_k\rangle \quad (\alpha = e, \mu, \tau)$$

The lepton charged current contains two parts, j^μ and $j^{\mu +}$

$$j^\mu = 2 \sum_\alpha \overline{\nu_{\alpha L}} \gamma^\mu l_{\alpha L} = 2 \sum_\alpha \sum_k U_{\alpha k}^* \overline{\nu_{kL}} \gamma^\mu l_{\alpha L}$$

is responsible for producing a ν_α

$$j^{\mu +} = 2 \sum_\alpha \overline{l_{\alpha L}} \gamma^\mu \nu_{\alpha L} = 2 \sum_\alpha \sum_k U_{\alpha k} \overline{l_{\alpha L}} \gamma^\mu \nu_{kL}$$

is responsible for producing a $\bar{\nu}_\alpha$

Therefore, one can easily obtain the expression

$$P_{\bar{\nu}_\alpha \to \bar{\nu}_\beta}(L,E) = \sum_{k,j} U_{\alpha k} U_{\beta k}^{*} U_{\alpha j}^{*} U_{\beta j} \exp\left(-i \frac{\Delta m_{kj}^2 L}{2E}\right)$$

and

$$P_{\bar{\nu}_\alpha \to \bar{\nu}_\beta}(L,E) = \delta_{\alpha\beta} - 4 \sum_{k>j} \mathrm{Re}\left[U_{\alpha k}^{*} U_{\beta k} U_{\alpha j} U_{\beta j}^{*}\right] \sin^2\left(\frac{\Delta m_{kj}^2 L}{4E}\right)$$

$$- 2 \sum_{k>j} \mathrm{Im}\left[U_{\alpha k}^{*} U_{\beta k} U_{\alpha j} U_{\beta j}^{*}\right] \sin\left(\frac{\Delta m_{kj}^2 L}{2E}\right)$$

under CP transformation, neutrino and antineutrino are interchanged

$$\nu_\alpha \xleftrightarrow{CP} \bar{\nu}_\alpha$$

Therefore

$$\nu_\alpha \to \nu_\beta \xleftrightarrow{CP} \bar{\nu}_\alpha \to \bar{\nu}_\beta$$

For time-reversal

$$\nu_\alpha \to \nu_\beta \xleftrightarrow{T} \nu_\beta \to \nu_\alpha$$

or

$$\bar{\nu}_\alpha \to \bar{\nu}_\beta \xleftrightarrow{T} \bar{\nu}_\beta \to \bar{\nu}_\alpha$$

Combining CP and T, we have CPT

$$\nu_\alpha \to \nu_\beta \xleftrightarrow{CPT} \bar{\nu}_\beta \to \bar{\nu}_\alpha$$

From

$$P_{\nu_\alpha \to \nu_\beta} = \sum_{k,j} U_{\alpha k}^* U_{\beta k} U_{\alpha j} U_{\beta j}^* \exp\left[-i \frac{\Delta m_{kj}^2 L}{2E}\right]$$

and

$$P_{\bar{\nu}_\alpha \to \bar{\nu}_\beta} = \sum_{k,j} U_{\alpha k} U_{\beta k}^* U_{\alpha j}^* U_{\beta j} \exp\left[-i \frac{\Delta m_{kj}^2 L}{2E}\right].$$

One easily obtain

$$P_{\nu_\alpha \to \nu_\beta} = P_{\bar{\nu}_\beta \to \bar{\nu}_\alpha} \qquad \text{(CPT invariance)}$$

(by interchanging α, β in $P_{\bar{\nu}_\alpha \to \bar{\nu}_\beta}$, for example)

A consequence is that

$$P_{\nu_\alpha \to \nu_\alpha} = P_{\bar{\nu}_\alpha \to \bar{\nu}_\alpha}$$

(survival probality for ν_α and $\bar{\nu}_\alpha$ identical under CPT)

CPT violation is reflected in a non-zero CPT asymmetry

$$A_{\alpha\beta}^{CPT} = P_{\nu_\alpha \to \nu_\beta} - P_{\bar{\nu}_\beta \to \bar{\nu}_\alpha} \neq 0$$

From

$$P_{\nu_\alpha \to \nu_\beta} = \delta_{\alpha\beta} - 4 \sum_{k>j} Re\left[U_{\alpha k}^* U_{\beta k} U_{\alpha j} U_{\beta j}^* \right] \sin^2\left(\frac{\Delta m_{kj}^2 L}{4E} \right)$$
$$+ 2 \sum_{k>j} Im\left[U_{\alpha k}^* U_{\beta k} U_{\alpha j} U_{\beta j}^* \right] \sin\left(\frac{\Delta m_{kj}^2 L}{2E} \right)$$

and

$$P_{\bar{\nu}_\alpha \to \bar{\nu}_\beta} = \delta_{\alpha\beta} - 4 \sum_{k>j} Re\left[U_{\alpha k}^* U_{\beta k} U_{\alpha j} U_{\beta j}^* \right] \sin^2\left(\frac{\Delta m_{kj}^2 L}{4E} \right)$$
$$- 2 \sum_{k>j} Im\left[U_{\alpha k}^* U_{\beta k} U_{\alpha j} U_{\beta j}^* \right] \sin\left(\frac{\Delta m_{kj}^2 L}{2E} \right)$$

We first see that CP violation implies a non-zero value for

$$A_{\alpha\beta}^{CP} = P_{\nu_\alpha \to \nu_\beta} - P_{\bar{\nu}_\alpha \to \bar{\nu}_\beta} \neq 0$$

but $P_{\nu_\alpha \to \nu_\beta} = P_{\bar{\nu}_\beta \to \bar{\nu}_\alpha}$ and $P_{\bar{\nu}_\alpha \to \bar{\nu}_\beta} = P_{\nu_\beta \to \nu_\alpha}$ for CPT invariance.

Hence

$$A_{\alpha\beta}^{CP} = P_{\nu_\alpha \to \nu_\beta} - P_{\bar{\nu}_\alpha \to \bar{\nu}_\beta} = P_{\bar{\nu}_\beta \to \bar{\nu}_\alpha} - P_{\nu_\beta \to \nu_\alpha}$$
$$= -\left(P_{\nu_\beta \to \nu_\alpha} - P_{\bar{\nu}_\beta \to \bar{\nu}_\alpha} \right) = - A_{\beta\alpha}^{CP}$$

This implies

$$A_{\alpha\alpha}^{CP} = - A_{\alpha\alpha}^{CP} = 0$$

Hence, there is no CP violation $\overbrace{}^{\text{(asymmetry)}}$ in disappearance experiments

This can also be easily seen from the expressions for $P_{\nu_\alpha \to \nu_\beta}$ and $P_{\bar{\nu}_\alpha \to \bar{\nu}_\beta}$

$$A^{CP}_{\alpha\beta} = 4 \sum_{k>j} \text{Im}\left[U^*_{\alpha k} U_{\beta k} U_{\alpha j} U^*_{\beta j} \right] \sin\left(\frac{\Delta m^2_{kj} L}{2E} \right)$$

If $\alpha = \beta$,

then $U^*_{\alpha k} U_{\beta k} U_{\alpha j} U^*_{\beta j} = |U_{\alpha k}|^2 |U_{\alpha j}|^2$

is a real quantity

Hence $A^{CP}_{\alpha\beta} = 0$ for $\alpha = \beta$ (since it would have violated CPT)

Note that

$A^{CP}_{\alpha\beta}(L, E)$ is oscillatory in L and E.

Depending on the values of Δm^2_{kj} and the amount of smearing in L, and in E, the averaged $\langle A^{CP}_{\alpha\beta} \rangle$ could be greatly reduced.

For $N=2$ neutrino oscillation, all matrix elements U_{ij} are real, hence

$$\text{Im}\left[U^*_{\alpha k} U_{\beta k} U_{\alpha j} U^*_{\beta j} \right] = 0$$

and

$$A^{CP}_{\alpha\beta} = 0 \quad \text{for } N = 2$$

For CPT invariance, CP violation would imply T violation. One defines the T-violating asymmetry

$$A^{T}_{\alpha\beta} = P_{\nu_\alpha \to \nu_\beta} - P_{\nu_\beta \to \nu_\alpha}$$

CPT invariance implies $P_{\nu_\beta \to \nu_\alpha} = P_{\bar{\nu}_\alpha \to \bar{\nu}_\beta}$ ∴

Hence

$$A^{T}_{\alpha\beta} = P_{\nu_\alpha \to \nu_\beta} - P_{\nu_\beta \to \nu_\alpha} = P_{\nu_\alpha \to \nu_\beta} - P_{\bar{\nu}_\alpha \to \bar{\nu}_\beta}$$
$$= A^{CP}_{\alpha\beta}$$

Similarly

$$\bar{A}^{T}_{\alpha\beta} = P_{\bar{\nu}_\alpha \to \bar{\nu}_\beta} - P_{\bar{\nu}_\beta \to \bar{\nu}_\alpha}$$
$$= P_{\nu_\beta \to \nu_\alpha} - P_{\bar{\nu}_\beta \to \bar{\nu}_\alpha} = A^{CP}_{\beta\alpha} = -A^{CP}_{\alpha\beta}$$

Hence

$$A^{T}_{\alpha\beta} = A^{CT}_{\alpha\beta} = -\bar{A}^{T}_{\alpha\beta}$$

This expression can also be obtained directly from the expressions for $P_{\nu_\alpha \to \nu_\beta}$ and $P_{\bar{\nu}_\alpha \to \bar{\nu}_\beta}$

For the familiar case of two-neutrino oscillation

$$U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

The transition probability

$$P_{\nu_\alpha \to \nu_\beta} = \delta_{\alpha\beta} - 4 \sum_{k>j} Re\{ U_{\alpha k}^* U_{\beta k} U_{\alpha j} U_{\beta j}^*] \sin^2\left(\frac{\Delta m_{kj}^2 L}{4E}\right)$$
$$+ 2 \sum_{k>j} Im\{ U_{\alpha k}^* U_{\beta k} U_{\alpha j} U_{\beta j}^*] \sin^2\left(\frac{\Delta m_{kj}^2 L}{4E}\right)$$

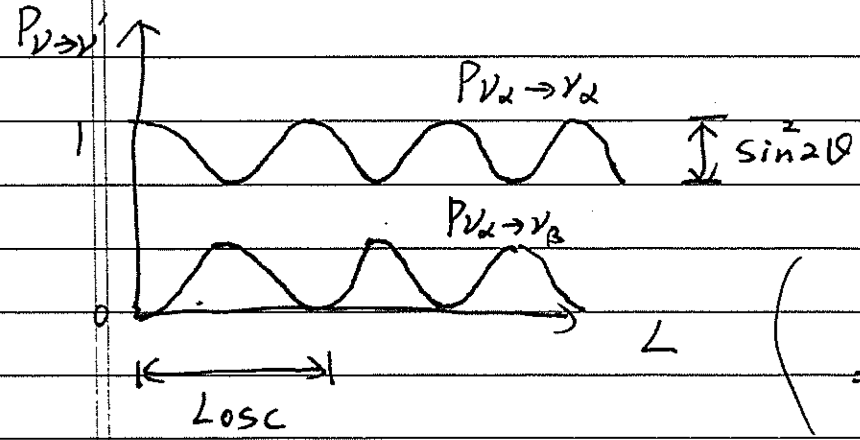
becomes $(U_{12}^* U_{22} U_{11}^* U_{21} = -\sin^2\theta \cos^2\theta$

$$P_{\nu_\alpha \to \nu_\beta} = (-4)(-\sin^2\theta\cos^2\theta)\sin^2\left(\frac{\Delta m^2 L}{4E}\right)$$
$$= \sin^2 2\theta \sin^2\left(\frac{\Delta m^2 L}{4E}\right) = \sin^2 2\theta \sin^2\left(\frac{\pi L}{L_{osc}}\right)$$

(if $\alpha \neq \beta$) $\qquad L_{osc} = \frac{4\pi E}{L}$

$$P_{\nu_\alpha \to \nu_\alpha} = 1 - \sin^2 2\theta \sin^2\left(\frac{\Delta m^2 L}{4E}\right)$$

$0 \leq \theta \leq \pi/2$



$$\left(\begin{array}{l} P_{\nu_\alpha \to \nu_\beta} = P_{\nu_\beta \to \nu_\alpha} \\ = P_{\bar\nu_\alpha \to \bar\nu_\beta} = P_{\bar\nu_\beta \to \bar\nu_\alpha} \end{array} \right)$$

Note that for a given value of $\sin^2 2\theta$, there are two solutions for θ, $0 < \theta \le \pi/2$

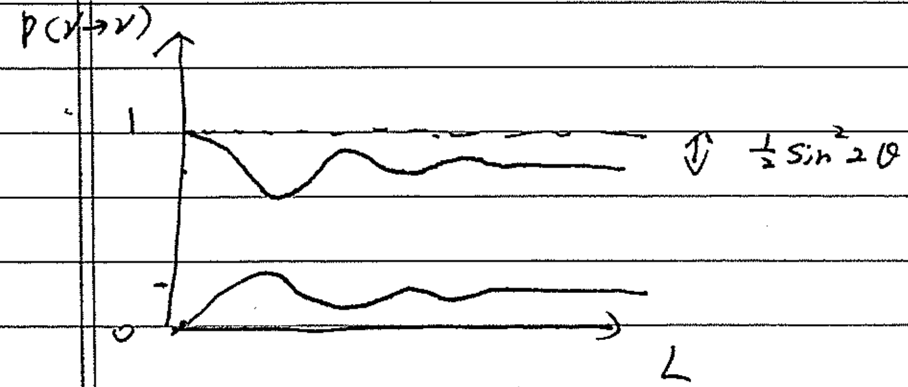
$$\sin^2 2\theta = 4\sin^2\theta\cos^2\theta = 4\sin^2\theta(1-\sin^2\theta) = a$$

$$4\sin^4\theta - 4\sin^2\theta + a = 0$$

$$\sin^2\theta = \frac{1 \pm \sqrt{1 - a/4}}{2}$$

two values for $\sin^2\theta$, or two solutions for $0 < \theta < \pi/2$

If one averages over E (with ΔE)



The average transition probability is

$$\langle P_{\nu_\alpha\to\nu_\beta}\rangle = \sum_k |U_{\alpha k}|^2 |U_{\beta k}|^2$$

For $N=2$, $\langle P_{\nu_\alpha\to\nu_\alpha}\rangle$ has a maximal value of $\frac{1}{2}$

For $N>2$, what is the maximal value of $\langle P_{\nu_\alpha\to\nu_\beta}\rangle$?

Using Lagrange multiplier

$$f = \sum_k |U_{\alpha k}|^2 |U_{\beta k}|^2 - a\left(1-\sum_k |U_{\alpha k}|^2\right) - b\left(1-\sum_k |U_{\beta k}|^2\right)$$

At the maximal value of f

$$\frac{\partial f}{\partial |U_{\alpha k}|^2} = 0 \Rightarrow |U_{\beta k}|^2 + a = 0 \quad \text{(for all } k\text{)}$$

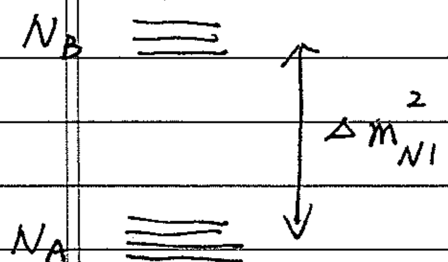
$$\frac{\partial f}{\partial |U_{\beta k}|^2} = 0 \Rightarrow |U_{\alpha k}|^2 + b = 0 \quad \text{(for all } k\text{)}$$

$$|U_{\alpha k}^{max}|^2 = |U_{\beta k}^{max}|^2 = \frac{1}{N} \quad \text{for all } k$$

$$\langle P_{\nu_\alpha\to\nu_\beta}\rangle_{max} = \sum_k |U_{\alpha k}^{max}|^2 |U_{\beta k}^{max}|^2 = \sum_{k=1,N}\left(\frac{1}{N}\right)\left(\frac{1}{N}\right) = \frac{1}{N}$$

The expression for $P_{\nu_\alpha \to \nu_\beta}(L, E)$ becomes very complicated when $N > 2$. However, significant simplification occurs for two situations

a) Dominance of a single Δm^2



$$N = N_A + N_B$$

$$|\Delta m^2_{N1}| \gg |\Delta m^2_{kj}|$$

if $k, j \le N_A$, or $k, j > N_A$

If the experiment is performed with L, E such that

$$\frac{|\Delta m^2_{N1}| L}{2E} \sim \pi \qquad (L \sim L_{osc}(\Delta m^2_{N1}))$$

then

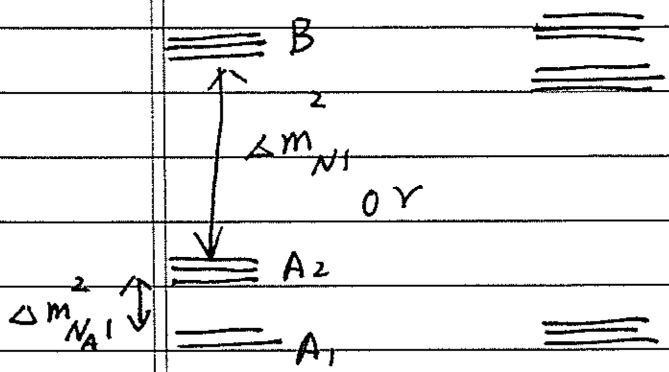
$$P_{\nu_\alpha \to \nu_\beta} \sim 4 \left| \sum_{k=1}^{N_A} U^*_{\alpha k} U_{\beta k} \right|^2 \sin^2\left(\frac{\Delta m^2_{N1} L}{4E}\right)$$

$$= \sin^2 2\theta^{eff}_{\alpha\beta} \sin^2\left(\frac{\Delta m^2_{N1} L}{4E}\right)$$

$$\sin^2 2\theta^{eff}_{\alpha\beta} = 4 \left| \sum_{k=1}^{N_A} U^*_{\alpha k} U_{\beta k} \right|^2 = 4 \left| \sum_{k=N_A+1}^{N} U^*_{\alpha k} U_{\beta k} \right|^2$$

Can not probe CP-violation

b) Dominance of a small Δm^2



$$N = N_{A_1} + N_{A_2} + N_B$$

For an experiment with L, E such that

$$\frac{|\Delta m^2_{NA1}|\, L}{2E} \sim \pi$$

$$P_{\nu_\alpha \to \nu_\alpha}(L,E) = 1 - \sin^2 2\theta^{eff}_{\alpha\alpha} \sin^2\left(\frac{\Delta m^2_{NA1} L}{4E}\right)$$

where

$$\cancel{\sin^2 2\theta^{eff}_{\alpha\alpha} = 4 \frac{\left(\sum_{k\le N_{A1}} |U_{\alpha k}|^2\right)}{\sum_{k=1}^{N_A} |U_{\alpha k}|^2}\left(1 - \frac{\sum_{k\le N_{A1}} |U_{\alpha k}|^2}{\sum_{k=1}^{N_A} |U_{\alpha k}|^2}\right)}$$

$$\sin^2 2\theta^{eff}_{\alpha\alpha} = 4 \frac{\left(\sum_{k\le N_{A1}} |U_{\alpha k}|^2\right)\left(\sum_{k=N_{A1}+1}^{N_A} |U_{\alpha k}|^2\right)}{\left(\sum_{k=1}^{N_A} |U_{\alpha k}|^2\right)^2}$$