Massive Neutrinos

In the Standard Model, neutrinos are assumed to be massless. Since we know now that neutrinos are massive, how do we modify the Standard Model to accommodate this fact?

We recall that the mass of electron in the SM is from the Yukawa coupling term

$$\mathcal{L}_{eY} = -C_e\left[\bar{\Psi}_R(\phi^+\Psi_L) + (\bar{\Psi}_L\phi)\Psi_R\right]$$

Since,

$$\bar{\Psi}_R(\phi^+\Psi_L) = \bar{e}_R\left(0, \frac{v+H}{\sqrt{2}}\right)\begin{pmatrix}\nu_L\\ e_L\end{pmatrix} = \frac{v+H}{\sqrt{2}}\,\bar{e}_R e_L$$

$$(\bar{\Psi}_L\phi)\Psi_R = \frac{v+H}{\sqrt{2}}\,\bar{e}_L e_R$$

we have

$$\mathcal{L}_{eY} = -\frac{C_e}{\sqrt{2}}(v\bar{e}e + H\bar{e}e), \quad \bar{e}e = \bar{e}_L e_R + \bar{e}_R e_L$$

The $v\bar{e}e$ term gives non-zero mass to electron.

Therefore, it seems that we can simply add ν_R as a new member to the $SU(2)_L \times U(1)_Y$, namely

$$SU(2)_L: \quad \Psi_L = \begin{pmatrix}\nu_L\\ e_L\end{pmatrix}; \quad e_R; \quad \nu_R$$

However, if one adds a new term to $\mathcal{L}_{eY}$ coming from ν_R, we would then have

$$\mathcal{L}'_{eY} = -C_\nu\left[\bar{\nu}_R(\phi^+\Psi_L) + (\bar{\Psi}_L\phi)\nu_R\right]$$

One would then obtain

$$\mathcal{L}_{eY}' = -\frac{C_\nu}{\sqrt{2}}(v+H)(\bar{\nu}_R e_L + \bar{e}_L \nu_R)$$

This does not make sense. The neutrino mass term should be proportional to $\bar{\nu}_R \nu_L$ and $\bar{\nu}_L \nu_R$

The solution is to use the charge-conjugate scalar field:

$$\phi^c = i\tau_2 \phi^* \quad \text{(analogous to spin-}\tfrac{1}{2}\text{ case)}$$

(recall $\psi^c(x) = \xi_F C \bar{\psi}^T(x) = \xi_F C \gamma_0^* \psi^*$, where $C = \gamma^2\gamma^0$, hence $\psi^c(x) = \xi_F \gamma^2 \psi^*$)

Hence $\phi^c = i\tau_2\phi^* = \begin{pmatrix} \bar{\phi}^0 \\ -\phi^- \end{pmatrix}$

For $\phi = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v+H \end{pmatrix}$, $\phi^c = \frac{1}{\sqrt{2}}\begin{pmatrix} v+H \\ 0 \end{pmatrix}$,

The correct expression for $\mathcal{L}_{eY}'$ is then

$$\mathcal{L}_{eY}' = -C_\nu \{\bar{\nu}_R (\phi^c)^\dagger \psi_L + (\bar{\psi}_L \phi^c)\nu_R]$$

and one obtains

$$\mathcal{L}_{eY}' = -\frac{C_\nu}{\sqrt{2}}(v+H)(\bar{\nu}_R \nu_L + \bar{\nu}_L \nu_R)$$

as one expects.

However, we still have a problem. We now know that electron neutrino mixes with other neutrinos to form the mass eigenstates. Therefore, electron neutrino does not have a definite mass, and it does not make sense to have a neutrino mass term such as

$$C_{\nu_e}\left(\overline{\nu_{eR}}\,\nu_{eL} + \overline{\nu_{eL}}\,\nu_{eR}\right)$$

In order to see how we can solve this problem, we need to go beyond the simple case of a single lepton family in the SM. Instead, we would consider three families, not only for the leptons, but also for the quarks, which also participate in the electroweak interaction:

$$\begin{pmatrix}\nu_e\\ e^-\end{pmatrix} \quad \begin{pmatrix}\nu_\mu\\ \mu^-\end{pmatrix} \quad \begin{pmatrix}\nu_\tau\\ \tau^-\end{pmatrix} \quad \begin{pmatrix}u\\ d\end{pmatrix} \quad \begin{pmatrix}c\\ s\end{pmatrix} \quad \begin{pmatrix}t\\ b\end{pmatrix}$$

We immediately see one apparent difference between the lepton sector and quark sector. For the lepton sector, the charged current only couples members in the same family namely,

$$\begin{pmatrix} \nu_e \\ \updownarrow \\ e^- \end{pmatrix} \quad \begin{pmatrix} \nu_\mu \\ \updownarrow \\ e^- \end{pmatrix} \quad \begin{pmatrix} \nu_\tau \\ \updownarrow \\ \tau^- \end{pmatrix}$$

However, for the quark sector, the charged current can couple any upper member to any lower member:

$$\begin{pmatrix} u \\ \updownarrow \\ d \end{pmatrix} \quad \begin{pmatrix} c \\ \updownarrow \\ s \end{pmatrix} \quad \begin{pmatrix} t \\ \updownarrow \\ b \end{pmatrix}$$

To address this apparent disparity between leptons and quarks, we now discuss the SM taking into account of all quarks and all leptons (but still assumes massless neutrinos, for now)

The $SU(2) \times U(1)$ classification and quantum numbers of the fundamental fermions in SM are as follows:

	1	2	3	I	I_3	Y	Q
l_{AL}	$\begin{pmatrix} \nu'_{eL} \\ e'_L \end{pmatrix}$	$\begin{pmatrix} \nu'_{\mu L} \\ \mu'_L \end{pmatrix}$	$\begin{pmatrix} \nu'_{\tau L} \\ \tau'_L \end{pmatrix}$	$\frac{1}{2}$	$\begin{pmatrix} +\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$	-1	$\begin{pmatrix} 0 \\ -1 \end{pmatrix}$
e'_{AR}	e'_R	μ'_R	τ'_R	0	0	-2	-1
q_{AL}	$\begin{pmatrix} u'_L \\ d'_L \end{pmatrix}$	$\begin{pmatrix} c'_L \\ s'_L \end{pmatrix}$	$\begin{pmatrix} t'_L \\ b'_L \end{pmatrix}$	$\frac{1}{2}$	$\begin{pmatrix} +\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$	$\frac{1}{3}$	$\begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}$
u'_{AR}	u'_R	c'_R	t'_R	0	0	$\frac{4}{3}$	$\frac{2}{3}$
d'_{AR}	d'_R	s'_R	b'_R	0	0	$-\frac{2}{3}$	$-\frac{1}{3}$

We distinguish the 'gauge symmetry' basis from the 'physical (mass)' basis.

The 'gauge symmetry' basis are eigenstates of the gauge symmetry, and are denoted as f'. We also use l_{AL} and q_{AL}, where $A=1,2,3$, to denote the three lepton and quark doublets (left-handed), and e'_{AR}, u'_{AR}, d'_{AR} for the lepton and quark singlets ($A=1,2,3$) (u refers to the upper members and d to the lower members of the doublets)

The gauge-invariant fermion Lagrangian becomes

$$
\begin{aligned}
\mathcal{L}_F = \; & \bar{l}_{AL}\, i\gamma^\mu D_\mu^L l_{AL} + \bar{e}'_{AR}\, i\gamma^\mu D_\mu^R e'_{AR} \\
& + \bar{q}_{AL}\, i\gamma^\mu D_\mu^L q_{AL} + \bar{u}'_{AR}\, i\gamma^\mu D_\mu^R u'_{AR} \\
& + \bar{d}'_{AR}\, i\gamma^\mu D_\mu^R d'_{AR}
\end{aligned}
$$

Notice the difference between lepton and quark sectors due to the massless (absence) of ν_{AR}.

If neutrino is massive, then an additional term must be added to $\mathcal{L}_F$, i.e.

$$\bar{\nu}'_{AR}\, i\gamma^\mu D_\mu^R \nu'_{AR}$$

The covariant derivatives are given as

$$D_\mu^L l_{AL} = (\partial_\mu + i \tfrac{1}{2} g \tau_i A_\mu^i - i \tfrac{1}{2} g' B_\mu) l_{AL}$$

$$D_\mu^R e'_{AR} = (\partial_\mu - i g' B_\mu) e'_{AR}$$

$$D_\mu^L q_{AL} = (\partial_\mu + i \tfrac{1}{2} g \tau_i A_\mu^i + i \tfrac{1}{6} g' B_\mu) q_{AL}$$

$$D_\mu^R u'_{AR} = (\partial_\mu + i \tfrac{2}{3} g' B_\mu) u'_{AR}$$

$$D_\mu^R d'_{AR} = (\partial_\mu - i \tfrac{1}{3} g' B_\mu) d'_{AR}$$

All covariant derivative contain the g' term from the $U(1)_Y$, but only the left-handed doublets contain the g term from $SU(2)_L$.

If ν_{AR} is included, then

$$D_\mu^R \nu'_{AR} = (\partial_\mu + i g' B_\mu \left(\tfrac{Y_{\nu_R}}{2}\right)) \nu'_{AR}$$

but $Y_{\nu_R} = [Q_{\nu_R} - (T_3)_{\nu_R}] 2 = 0$

hence $D_\mu^R \nu'_{AR} = \partial_\mu \nu'_{AR}$

and there is no $U(1)$ coupling for ν_R!

(ν_R does not couple to Z-boson)

The Yukawa coupling term becomes

$$\mathcal{L}_Y = -[C^e_{AB}(\bar{\ell}_{AL}\phi)e'_{BR} + C^u_{AB}(\bar{q}_{AL}\phi^c)u'_{BR} + C^d_{AB}(\bar{q}_{AL}\phi)d'_{BR}] + h.c.$$

A and B refer to the family, and $A, B = 1, 2, 3$.

It is important to note that C^e_{AB}, C^u_{AB}, C^d_{AB} are members of three different 3×3 matrices. They can have non-zero off-diagonal matrix elements, which allow the possibility (indeed, the reality) that the 'gauge symmetry' eigenstates can be different from the 'mass' eigenstates.

using $\phi = \begin{pmatrix} 0 \\ \frac{v+H}{\sqrt{2}} \end{pmatrix}$ and $\phi^c = \begin{pmatrix} \frac{v+H}{\sqrt{2}} \\ 0 \end{pmatrix}$

we obtain

$$\mathcal{L}_Y = -\left(1+\frac{H}{v}\right)(\bar{e}'_L M'_e e'_R + \bar{u}'_L M'_u u'_R + \bar{d}'_L M'_d d'_R) + h.c.$$

where M'_f are the fermionic mass matrices in the gauge eigenstate basis:

$$M'_f = \frac{v}{\sqrt{2}} C^f \qquad (f = e, u, d)$$

Note that there are two mass matrices for the quark sector, one for the upper members, M_u', and the other for the lower members, M_d'.
If neutrino is massive, then there would also be another mass matrix for neutrinos :

$$M_\nu' = \frac{v}{\sqrt{2}} C^\nu$$

The mass matrices are not required to be unitary, symmetric or Hermitian. However, they can be diagonalized by the 'biunitary' transformation as follows.

One can express M_f' as a product of a Hermitian matrix and a unitary matrix

$$M_f' = \underset{\substack{\uparrow \\ \text{Hermitian}}}{H_f} \; \underset{\substack{\uparrow \\ \text{unitary}}}{T_f}$$

then

$$M_f' M_f'^\dagger = (H_f T_f)(H_f T_f)^\dagger = |H_f|^2$$

$$H_f = \sqrt{|M_f'|^2}$$

H_f is Hermitian and positive definite, hence it can be diagonalized by a unitary matrix, S_f

$$S_f H_f S_f^\dagger = M_f$$

This means

$$M_f' = H_f T_f = S_f^{+} M_f S_f T_f, \quad \begin{pmatrix} M_f \text{ is a diagonal} \\ \text{mass matrix} \end{pmatrix}$$

and the mass term in $\mathcal{L}_Y$ becomes

$$\bar{\psi}_{fL}' M_f' \psi_{fR}' = \bar{\psi}_{fL} M_f \psi_{fR}, \quad f = e, u, d.$$

The 'mass eigenstates' ψ_i are related to the 'gauge eigenstates' ψ_i' by the unitary transformations

$$\psi_{fL} = S_f \psi_{fL}'$$

$$\psi_{fR} = S_f T_f \psi_{fR}' \qquad (S_f T_f \text{ is unitary})$$

$$\underset{\text{mass eigenstates}}{\uparrow} \qquad\qquad \underset{\text{gauge eigenstates}}{\uparrow}$$

The mass matrix M_f is diagonal

$$(M_f)_{AB} = m_A^f \delta_{AB}; \quad M_e = \begin{pmatrix} m_e & & 0 \\ & m_\mu & \\ 0 & & m_\tau \end{pmatrix}$$

Hence $\mathcal{L}_Y$ becomes $\quad M_u = \begin{pmatrix} m_u & & 0 \\ & m_c & \\ 0 & & m_t \end{pmatrix}; \quad M_d = \begin{pmatrix} m_d & & 0 \\ & m_s & \\ 0 & & m_b \end{pmatrix}$

$$\mathcal{L}_Y = -\left(1 + \frac{H}{v}\right)\left[m_A^e (\bar{e}_A e_A) + m_A^u (\bar{u}_A u_A) + m_A^d (\bar{d}_A d_A)\right]$$

(summed over $A = 1, 2, 3$)

Now consider the charged-current for quarks

$$J_\mu^\dagger = \overline{u}_L' \gamma_\mu d_L' \quad \text{(go back to the gauge basis)}$$

but

$$d_L = S_d d_L' \; ; \quad u_L = S_u u_L'$$

$$(d_L' = S_d^+ d_L \; ; \quad u_L' = S_u^+ u_L)$$

hence

$$J_\mu^+ = \overline{u}_L \gamma_\mu S_u S_d^+ d_L = \overline{u}_L \gamma_\mu d''_L$$

where $d'' = S_u S_d^+ d = V d$

V is the CKM matrix:

$$\begin{pmatrix} d'' \\ s'' \\ b'' \end{pmatrix} \doteq \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix}$$

Similarly, if neutrino is massive, the charged current for leptons becomes

$$J_\mu^+ = \overline{\nu}_L' \gamma_\mu e_L' = \overline{\nu}_L'' \gamma_\mu e_L$$

where $\overline{\nu}_L'' = \overline{\nu}_L S_\nu S_e^+$ or $\nu_L'' = S_e S_\nu^+ \nu_L$

Set $U = S_e S_\nu^+$

then

$$\begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} = U_{PMNS} \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix}$$

U_{PMNS} is the neutrino mixing matrix

If M_e' is diagonal (charged leptons mass eigenstates are the same as gauge eigenstates), then $S_e = I$, and

$$U_{PMNS} = S_\nu^\dagger$$

In order words, the mixing matrix is the "unitary" matrix, which can diagonalize the 'arbitrary' mass matrix

$$S_\nu M_\nu' S_\nu^\dagger = M_\nu$$

$$U^\dagger = R_{23}\, R_{13}\, R_{12}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & C_{23} & S_{23} \\ 0 & -S_{23} & C_{23} \end{pmatrix} \begin{pmatrix} C_{13} & 0 & S_{13} e^{-i\delta} \\ 0 & 1 & 0 \\ -S_{13} e^{i\delta} & 0 & C_{13} \end{pmatrix} \begin{pmatrix} C_{12} & S_{12} & 0 \\ -S_{12} & C_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$C_{ij} = \cos\theta_{ij} \quad ; \quad S_{ij} = \sin\theta_{ij}$$

$$= \begin{pmatrix} C_{12} C_{13} & S_{12} C_{13} & S_{13} e^{-i\delta} \\ -S_{12} C_{23} - S_{13} S_{23} C_{12} e^{i\delta} & C_{12} C_{23} - S_{12} S_{13} S_{23} e^{i\delta} & S_{23} C_{13} \\ S_{12} S_{23} - S_{13} C_{12} C_{23} e^{i\delta} & -S_{23} C_{12} - S_{12} S_{13} C_{23} e^{i\delta} & C_{13} C_{23} \end{pmatrix}$$

For the quark sector, $\theta_{12} = 12.9^\circ$ (cabbibo angle)

$\theta_{23} = 2.4^\circ$, $\theta_{13} = 0.2^\circ$, $\delta = 59^\circ \pm 13^\circ$

For the lepton sector, $\theta_{12} \sim 33.6^\circ$, $\theta_{23} \sim 45^\circ$, $\theta_{13} \sim 8.$

$S_{12} = \sqrt{\frac{1}{3}}$, $C_{12} = \sqrt{\frac{2}{3}}$, $S_{23} = C_{23} = \frac{1}{\sqrt{2}}$ $\delta = ?$

$S_{13} = 0$, $C_{13} = 1$

$$U_{PMNS} \sim \begin{pmatrix} \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} & 0 \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{2}} \\ \sqrt{\frac{1}{6}} & -\sqrt{\frac{1}{3}} & \sqrt{\frac{1}{2}} \end{pmatrix}$$

Mixing angles $\theta_{12}, \theta_{23}, \theta_{13}$

and CP phase ~~are not "physical"~~

(they depend on the parametrization)

$U = R_{23} \; R_{13} \; R_{12}$ conventional

$U = R_{23} \; R_{12} \; R_{13}$ or any other permutation.

The values of $\theta_{12}, \theta_{23}, \theta_{13}$ and δ_{CP} ~~would change, if one uses a different~~ order for matrices multiplication

However, the mixing matrix U has the ~~same matrix elements, independent of the~~ multiplication orders.

The conventional choice of the mixing angles is very fortunate. It has the effects that $\theta_{12}, \theta_{23}, \theta_{13}$ can be determined, respectively, in the solar, atmospheric, and reactor neutrino oscillation experiments (to be discussed later).

$$\nu_e = \sqrt{\tfrac{2}{3}}\,\nu_1 + \sqrt{\tfrac{1}{3}}\,\nu_2$$

$$\nu_\mu = -\sqrt{\tfrac{1}{6}}\,\nu_1 + \sqrt{\tfrac{1}{3}}\,\nu_2 + \sqrt{\tfrac{1}{2}}\,\nu_3$$

$$\nu_\tau = \sqrt{\tfrac{1}{6}}\,\nu_1 - \sqrt{\tfrac{1}{3}}\,\nu_2 + \sqrt{\tfrac{1}{2}}\,\nu_3$$

$$\nu_1 = \sqrt{\tfrac{2}{3}}\,\nu_e - \sqrt{\tfrac{1}{6}}\,\nu_\mu + \sqrt{\tfrac{1}{6}}\,\nu_\tau$$

$$\nu_2 = \sqrt{\tfrac{1}{3}}\,\nu_e + \sqrt{\tfrac{1}{3}}\,\nu_\mu - \sqrt{\tfrac{1}{3}}\,\nu_\tau$$

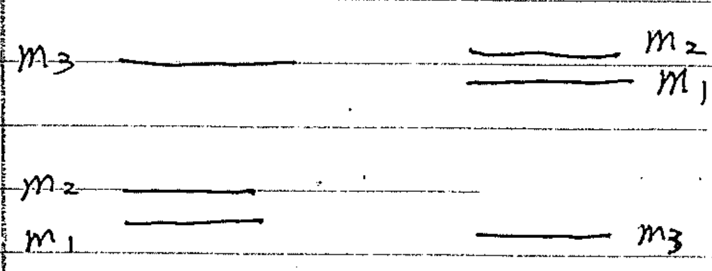
$$\nu_3 = \sqrt{\tfrac{1}{2}}\,\nu_\mu + \sqrt{\tfrac{1}{2}}\,\nu_\tau$$

From neutrino oscillation experiments

$$m_2^2 - m_1^2 = 7.5 \times 10^{-5}\ \text{eV}^2$$

(Solar neutrino)

$$\left| m_3^2 - m_1^2 \right| = 2.5 \times 10^{-3}\ \text{eV}^2$$



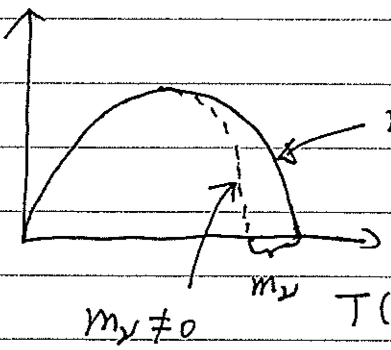
Normal Mass Hierarchy

Inverted Mass Hierarchy

Direct measurements of 'neutrino masses'

a) Electron neutrino

$${}^{3}_{1}H \rightarrow {}^{3}_{2}He + e^{-} + \bar{\nu}_e$$



18.6 keV endpoint energy

~~$m(\nu_e) < 2$ eV~~

Latest result from ~~KATRIN~~

Karlsruhe, Germany

Sept. 13, 2019, $\boxed{\text{ArXiv}: 1909.06048}$

~~1 month of data-taking~~

$\Rightarrow \quad m(\nu_e) < 1.1$ eV

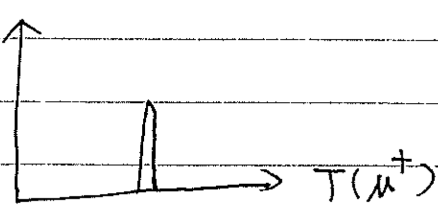
(hope to reach a sensitivity of ~ 0.2 eV

b) Muon neutrino

$$\pi^+ \longrightarrow \mu^+ + \nu_\mu$$

$$m_{\nu_\mu}^2 = m_\pi^2 + m_\mu^2 - 2\, m_\pi E_\mu$$

$$E_\mu = m_\mu + T_\mu$$



$m(\nu_\mu) < 0.19$ MeV

c) Tau neutrino

$$\tau^- \to \pi^+ \pi^- \pi^+ \nu_\tau$$

Use the "missing mass" ~~technique~~

$$p(\nu_\tau) = p(\tau^-) - p(\pi^+) - p(\pi^-) - p(\pi^{+\prime})$$

$$\cancel{p^2(\nu_\tau) = m^2(\nu_\tau)}$$

$$m(\nu_\tau) < 18.2 \text{ MeV}$$

Note that the above measurements set limits on the flavor eigenstates, which are combinations of the mass eigenstates.

Consider the ν_e,

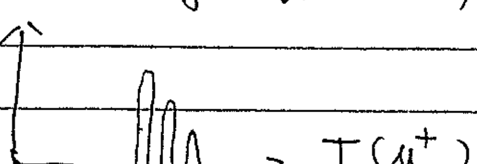
$$\nu_e = \sqrt{\tfrac{2}{3}}\,\nu_1 + \sqrt{\tfrac{1}{3}}\,\nu_2$$

~~Hence, the limit of~~ $m(\nu_e) < 1$ eV

~~implies~~ $m(\nu_1) < 1.5$ eV, $m(\nu_2) < 3$ eV

Also, in the $\pi^+ \to \mu^+ + \nu_\mu$ experiment, with ~~perfect accuracy, one can, in principle, observe~~

$$\nu_\mu = \sqrt{\tfrac{1}{6}}\,\nu_1 + \sqrt{\tfrac{1}{3}}\,\nu_2 + \sqrt{\tfrac{1}{2}}\,\nu_3$$



$$\begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} = U^{\dagger}_{PMNS} \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix}$$

$$U_{PMNS} = S_\nu^{\dagger}$$

where

$$S_\nu M_\nu' S_\nu^{\dagger} = M_\nu$$

The form of M_ν' mass matrix would determine U_{PMNS} mixing matrix $\Longrightarrow \theta_{12}, \theta_{23}, \theta_{13}$ will be determined

Also the diagonalized mass matrix would give the values of m_1, m_2, m_3

hence, the mass hierarchy is also determined

$\theta_{12} \sim 33.6^\circ$, $\theta_{23} \sim 45^\circ$, $\theta_{13} \sim 8.4^\circ$

$\delta_{CP} = ?$

$\Delta m_{21}^2 = 7.5 \times 10^{-5}\, eV^2$, $|\Delta m_{31}^2| = 2.5 \times 10^{-3}\, eV^2$

Modeling the Neutrino Mass Matrix

$$U_l^\dagger M_l U_l = \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix}; \quad U_\nu^\dagger M_\nu U_\nu = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}$$

$$U_{PMNS} = U_l^\dagger U_\nu$$

Examples of neutrino mass matrix:

$a)$ $\nu_\mu - \nu_\tau$ symmetry

$$M_\nu' = \begin{pmatrix} \overset{M_{ee}}{A} & \overset{M_{e\mu}}{B} & \overset{M_{e\tau}}{B} \\ \overset{M_{\mu e}}{B} & \overset{M_{\mu\mu}}{C} & \overset{M_{\mu\tau}}{D} \\ \overset{M_{\tau e}}{B} & \overset{M_{\tau\mu}}{D} & \overset{M_{\tau\tau}}{C} \end{pmatrix};$$

$\mu \leftrightarrow \tau$

$M_{\mu\mu} = M_{\tau\tau}$

$M_{e\mu} = M_{e\tau}$

$M_{\mu\tau} = M_{\tau\mu}$ $\quad M_{\mu e} = M_{\tau e}$

Symmetric with respect to $\nu_\mu - \nu_\tau$ permutation (diagonal M_l mass matrix)

After diagonalization, one can determine the mass hierarchy and the mixing parameters

– Normal mass hierachy ($m_1 < m_2 < m_3$)

– $\theta_{13} = 0,\ \theta_{23} = \pi/4$

Two ways to get $\theta_{13} \neq 0$:

$a)$ explicit symmetry breaking at the model scale

b) Radiative correction (RGE) from high scale to low scale

Modeling the Neutrino Mass Matrix

Examples of neutrino mass matrix:

b) ν mass matrices with texture zeros

(Frampton, Glashow, Marfatia)

$$M_\nu = \begin{pmatrix} 0 & 0 & X \\ 0 & X & X \\ X & X & X \end{pmatrix}; \qquad X : \text{non-zero entries}$$

$$\sin^2\theta_{13} \simeq \frac{R_\nu \tan^2\theta_{12}}{\tan^2\theta_{23}\,|1-\tan^4\theta_{12}|} \;\; ; \;\; R_\nu = \frac{\Delta m^2_{21}}{\Delta m^2_{31}}$$

$$M_\nu = \begin{pmatrix} 0 & X & 0 \\ X & X & X \\ 0 & X & X \end{pmatrix}; \qquad X : \text{non-zero entries}$$

$$\sin^2\theta_{13} \simeq \frac{R_\nu \tan^2\theta_{12}\tan^2\theta_{23}}{|1-\tan^4\theta_{12}|}$$

$$J_\mu^+ = \bar{\nu}_L U^+ \gamma_\mu e_L$$

where $e_L = \begin{pmatrix} e_L \\ \mu_L \\ \tau_L \end{pmatrix}$ $\quad \nu_L = \begin{pmatrix} \nu_{1L} \\ \nu_{2L} \\ \nu_{3L} \end{pmatrix}$.

both are mass eigenstates (fields)

$J_\mu^+ = \nu_L'' \gamma_\mu e_L$, where $\nu_L'' = \begin{pmatrix} \nu_{eL} \\ \nu_{\mu L} \\ \nu_{\tau L} \end{pmatrix}$

are the flavor fields

(note that $\nu_L' = \begin{pmatrix} \nu_{eL}' \\ \nu_{\mu L}' \\ \nu_{\tau L}' \end{pmatrix}$ are the

gauge-symmetry field)

why do we only have 3 mixing angles

$\theta_{12}, \theta_{13}, \theta_{23}$ and one CP phase for U?

For an $N \times N$ complex matrix, we have $2N^2$ parameters. The requirement of unitarity gives N^2 equations, and $2N^2 - N^2 = N^2$ gives N^2 remaining parameters:

$\frac{N(N-1)}{2}$ correspond to mixing angles

and $\frac{N(N+1)}{2}$ phases remain

Consider the case for $N = 3$

We can remove some phases by performing global $U(1)$ phase transformation :

$$e_{\alpha L} \rightarrow e^{i\phi_\alpha} e_{\alpha L}$$

$$\nu_{iL} \rightarrow e^{i\phi_i} e_{iL}$$

or
$$\begin{pmatrix} e_L \\ \mu_L \\ \tau_L \end{pmatrix} \rightarrow \begin{pmatrix} e^{i\phi_e} e_L \\ e^{i\phi_\mu} \mu_L \\ e^{i\phi_\tau} \tau_L \end{pmatrix}$$

$$\begin{pmatrix} \nu_{1L} \\ \nu_{2L} \\ \nu_{3L} \end{pmatrix} \rightarrow \begin{pmatrix} e^{i\phi_1} \nu_{1L} \\ e^{i\phi_2} \nu_{2L} \\ e^{i\phi_3} \psi_{3L} \end{pmatrix}$$

We have six independent phases at our disposal. However, we could only remove 5 phases from U^+. More specifically, after $U(1)$ transformation

$$J_\mu^+ = \sum_{i=1,2,3} \sum_{\alpha=e,\mu,\tau} \overline{\nu_{iL}} \gamma_\mu e^{-i\phi_i} U^+_{i\alpha} e^{i\phi_\alpha} e_{\alpha L}$$

$$= \underbrace{e^{-i(\phi_1-\phi_e)}}_{1 \text{ phase}} \sum_{i=1,2,3} \sum_{\alpha=e,\mu\tau} \overline{\nu_{iL}} \gamma_\mu \underbrace{e^{-i(\phi_i-\phi_1)}}_{2\text{phases}} U^+_{i\alpha} \underbrace{e^{i(\phi_\alpha-\phi_e)}}_{2\text{phases}} e_{\alpha L}$$

It is clear that we have only 5 independent ϕ phases at our disposal for reducing the number of phases in U^+.

Therefore,

$$2N^2 - N^2 - (2N-1)$$

$$= N^2 - 2N + 1 = \underbrace{\frac{N(N-1)}{2}}_{\text{\# of mixing angles}} + \underbrace{\frac{(N-1)(N-2)}{2}}_{\text{\# of phases}}$$

For $N=2$: 1 mixing angle, 0 phase

$N=3$: 3 mixing angles, 1 phase

Why are there 2 additional 'Majorana phases' if neutrinos are Majorana particles? We will discuss the topics of Majorana neutrinos later, but very briefly, the reason is that Majorana neutrinos are not invariant under global U(1) transformations :

Majorana field : $\nu = \nu_L + \nu_L^c$

under global U(1) transformation

$$\nu_L \to e^{i\phi}\nu_L \quad , \quad \nu_L^c \to e^{-i\phi}\nu_L^c$$

Therefore, one could only re-phase the charged-lepton fields (3 phases at our disposal), but not the Majorana neutrino fields. Hence, for $N=3$ we can only reduce 3 phases, rather than 5.

The 2 additional phases are the Majorana phases

We also need to consider the EM and neutral currents for the lepton sectors.

$$J_\mu^{em} = \bar{e}' \gamma_\mu Q e'$$

$$J_\mu^{Z} = \bar{\nu}_L' \gamma_\mu Z \nu_L' + \bar{e}' \gamma_\mu Z e'$$

Just like the charged current, the EM and neutral currents are expressed in terms of the gauge-symmetry fields e', ν_L'.

Do we need to worry about CKM-like or PMNS-like matrices for these currents?

The answer is No! The reason is that the field and adjoint field correspond to the same fermion, and unitarity of transformations V_L^f and V_R^f, where

$$\underset{\uparrow \text{ mass fields}}{\psi_{fL}} = V_L^{f\dagger} \underset{\uparrow \text{ gauge fields}}{\psi_{fL}'} \qquad V_L^f V_L^{f\dagger} = 1$$

$$\psi_{fR} = V_R^{f\dagger} \psi_{fR}' \qquad V_R^f V_R^{f\dagger} = 1$$

implies $J_\mu^{em} = \bar{e} \gamma_\mu Q e$

and $J_\mu^{Z} = \bar{\nu}_L \gamma_\mu Z \nu_L + \bar{e} \gamma_\mu Z e$

(No need to define e'' or ν'')