Lecture 5

Reminder: HW 1 due 2/13
Office hours have started

Recap: $\rho: G \to U(V)$ is an irreducible representation (irrep) if it has no nontrivial invariant subspaces ← we can't block diagonalize all $\rho(g)$ simultaneously

Schur's lemma: $\rho_1: G \to U(V_1)$ both irreducible
$\rho_2: G \to U(V_2)$

$H: V_1 \to V_2$ with $H\rho_1(g) = \rho_2(g)H$ for all $g \in G$

Then: (1) either $H = 0$ or H is invertible

(2) if $\rho_1 = \rho_2 = \rho$ are the same

$[H, \rho(g)] = 0$ and then

either $H = 0$ or $H = \lambda \, Id_V$

(2.5) if $H\rho_1(g) = \rho_2(g)H$ and if $H \neq 0$

then ρ_1 & ρ_2 are unitarily equivalent

Schur's lemma in QM:

Hamiltonian H with symmetry group G

$\rho : G \to U(\mathcal{H})$ $\qquad$ $\mathcal{H}$ - Hilbert space of wavefunction

$[\rho(g), H] = 0$ since g is a symmetry

Now suppose we focus on $V = \{|\psi_i\rangle, i = 1, \ldots N\}$

invariant subspace

$$\rho(g)|\psi_i\rangle = \sum_{j=1}^{N} |\psi_j\rangle \underbrace{\langle\psi_j|\rho(g)|\psi_i\rangle}_{\rho_{ji}(g)}$$

suppose $\{\rho_{ji}(g) \mid g \in G\}$ form an irreducible representation

$$[H]_{ij} = \langle\psi_i|H|\psi_j\rangle$$ - matrix elements of H

$$[\rho(g), H] \rightarrow \sum_{k=1}^{N} [H]_{ik}\,\rho_{kj}(g) - \rho_{ik}(g)\,[H]_{kj} = 0$$

Schur's lemma $\quad [H]_{ij} = E\,\delta_{ij}$

$\boxed{\text{States that transform in an irrep of the symmetry group are degenerate}}$

Example: Nonrelativistic Hydrogen atom

Symmetries of the Hamiltonian: SO(3) rotation invariant

$V_1 = \{|n l_1 m_z\rangle \mid m_z = -l_1, \ldots, l_1\}$ ← spin-l_1 irrep

$V_2 = \{|n l_2 m_z'\rangle \mid m_z' = -l_2, \ldots, l_2\}$ ← spin-l_2 irrep

non-square matrices are not invertible

Schur's lemma part 1

$$H_{m_z m_z'} = \langle n l_1 m_z | H | n l_2 m_z' \rangle = \begin{cases} l_1 \neq l_2 \Rightarrow H = 0 \\ l_1 = l_2 \Rightarrow E_{nl_1} \delta_{m_z m_z'} \end{cases}$$

Schur's lemma E_{nl_1} don't depend on m_z

(part 2.5: if $H \neq 0$ then $\rho_1 \cong \rho_2$
$\Leftrightarrow$ $\rho_1 \ncong \rho_2$ then $H = 0$)

Character theory:
- easy way to tell when representations are unitarily equivalent
- tell when representations are irreducible
- count & enumerate all irreps of a group

let G be a group, $\rho: G \to U(V)$ a finite-dimensional representation of G.

Def: the <u>character</u> χ_ρ, a function

$$\chi_\rho : G \to \mathbb{C}$$
$$\chi_\rho(g) = \mathrm{tr}[\rho(g)]$$

We get 3 useful properties

① $tr[U\rho(g)U^\dagger] = tr[U^\dagger U\rho(g)] = tr[\rho(g)]$ if U is unitary

$\Rightarrow$ if ρ_1 & ρ_2 $(\rho_1(g) = U\rho_2(g)U^\dagger$ for all $g)$

then $\chi_{\rho_1} = \chi_{\rho_2}$

equivalently if $\chi_{\rho_1} \neq \chi_{\rho_2}$ then $\rho_1 \neq \rho_2$

② say g_1 and g_2 are conjugate, $g_1 = g g_2 g^{-1}$ for some $g \in G$

$$\chi_\rho(g_1) = \mathrm{tr}[\rho(g_1)] = \mathrm{tr}[\rho(g g_2 g^{-1})]$$
$$= \mathrm{tr}[\rho(g)\rho(g_2)\rho^{-1}(g)]$$
$$= \mathrm{tr}[\rho(g_2)] = \chi_\rho(g_2)$$

characters are invariant under conjugation

all elements of the same conjugacy class

$$C_{g_1} = \{g' \in G \mid g' = g_2 g g_2^{-1}\}$$

have the same value $\chi_\rho(g_1) = \chi_\rho(g')$

→ Characters are constant on conjugacy classes

(we call these "class functions")

(3) If $\rho \cong \rho_1 \oplus \rho_2$

then $\chi_\rho = \chi_{\rho_1 \oplus \rho_2}$

but $[\rho_1 \oplus \rho_2](g) = \left(\begin{array}{c|c} \rho_1(g) & 0 \\ \hline 0 & \rho_2(g) \end{array}\right)$

$$\chi_{\rho_1 \oplus \rho_2}(g) = tr[\rho_1(g)] + tr[\rho_2(g)] = \chi_{\rho_1}(g) + \chi_{\rho_2}(g)$$

The character of a reducible representation is the sum of characters of its irreducible subrepresentations

Example: the group D_2 from HW1

$$D_2 = \{ E, C_{2x}, C_{2y}, C_{2z} \}$$

$$C_{2x}^2 = C_{2y}^2 = C_{2z}^2 = E$$

$$C_{2x} C_{2y} = C_{2y} C_{2x} = C_{2z}$$

vector representation ρ_v

$$\rho_v(E) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \rho_v(C_{2y}) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\rho_v(C_{2x}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad \rho_v(C_{2z}) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

abelian group $g_1 g_2 = g_2 g_1$ for all $g_1, g_2 \in G$

$g_1 g_2 g_1^{-1} = g_2$ for all $g_1, g_2 \in G$

$\Rightarrow$ each group element is in its own conjugacy class]

	E	C_{2x}	C_{2y}	C_{2z}
χ_{ρ_V}	3	-1	-1	-1

Invariant subspaces: $\left\{\begin{pmatrix} b_1 \\ 0 \\ 0 \end{pmatrix}\right\}, \left\{\begin{pmatrix} 0 \\ b_2 \\ 0 \end{pmatrix}\right\}, \left\{\begin{pmatrix} 0 \\ 0 \\ b_3 \end{pmatrix}\right\}$

$$\rho_V = \rho_{B_1} \oplus \rho_{B_2} \oplus \rho_{B_3}$$

[Aside: 1d representation]

$\rho(g)$ is 1×1

$tr(\rho(g)) = \rho(g)$

conjugacy classes

	E	C_{2x}	C_{2y}	C_{2z}
B_1	1	1	-1	-1
B_2	1	-1	1	-1
B_3	1	-1	-1	1

← Character table for the group

irreducible representations

One last general property

$$\chi_\rho(E) = tr(\rho(E)) = tr(Id_V) = \dim V = \dim \rho$$

Schur's Lemma $\Rightarrow$ Schur Orthogonality Relations
("Wonderful orthogonality theorem")

Let G be a finite group $\rho_1 : G \to U(V_1)$
$\rho_2 : G \to U(V_2)$
irreducible representations

and A <u>any</u> arbitrary $\dim V_2 \times \dim V_1$

$A : V_1 \to V_2$

trick: average over group elements

$$A_G = \sum_{g \in G} \rho_2(g^{-1}) A \rho_1(g)$$

$$A_G \rho_1(g') = \sum_{g \in G} \rho_2(g^{-1}) A \rho_1(g) \rho_1(g')$$

$$= \sum_{g \in G} \rho_2(\underbrace{g^{-1}}) A \rho_1(gg')$$

$$= \sum_{g'' \in G} \underbrace{\rho_2(g' g''^{-1})} A \rho_1(g'')$$

$g'' = gg'$

$g^{-1} = (g'' g'^{-1})^{-1}$

$= (g')(g'')^{-1}$

$$= \rho_2(g^{-1}) A_G$$

A_G satisfies the assumptions of Schur's lemma

$$A_G = \begin{cases} A_G = 0 & \rho_1 \neq \rho_2 \\ A_G = \lambda \mathrm{Id} & \rho_1 = \rho_2 \end{cases}$$

Lets apply this to a special choice

$$A = E_{ij} = \begin{array}{cc} \text{jth column} & \\ \begin{pmatrix} 0, & 0 & \cdots & \\ 0 & & 1 & \\ \vdots & & & \end{pmatrix} & \downarrow \text{ith row} \end{array}$$

$$[E_{ij}]^{\mu\nu} = \delta_{i\mu}\delta_{j\nu}$$

$$[E_{ij}]_G^{\mu\nu} = \sum_{g\in G}\sum_{\alpha\beta} [\rho_2(g^{-1})]^{\mu\alpha}[E_{ij}]^{\alpha\beta}[\rho_1(g)]^{\beta\nu}$$

$$= \sum_{g\in G} [\rho_2(g^{-1})]^{\mu i}[\rho_1(g)]^{j\nu} = \begin{cases} 0 & \text{if } \rho_1 \not\cong \rho_2 \\ \lambda\delta_{\mu\nu} & \text{if } \rho_1 = \rho_2 \end{cases}$$

To find λ, let $\rho_1 = \rho_2$ and take a trace

RHS: $\sum_{\mu} \lambda \delta_{\mu\mu} = \lambda \dim \rho_1$

LHS: $\sum_{g \in G} \sum_{\mu} [\rho_1(g^{-1})]^{\mu i} [\rho_1(g)]^{j\mu}$

$$\sum_{g \in G} \delta_{ij} = |G| \delta_{ij}$$

$$\lambda = \frac{|G|}{\dim \rho_1} \delta_{ij}$$

Schur Orthogonality Relation:

ρ_1, ρ_2 are irreps of a finite group then

$$\sum_{g\in G} [\rho_2(g^{-1})]^{\mu i} [\rho_1(g)]^{j\nu} = \begin{cases} 0 & \text{if } \rho_1 \not\cong \rho_2 \\ \frac{|G|}{\dim \rho_1} \delta_{ij} \delta_{\mu\nu} & \text{if } \rho_1 = \rho_2 \end{cases}$$

$$\sum_{g\in G} \sum_{\mu,\nu} [\rho_2(g^{-1})]^{\mu\mu} [\rho_1(g)]^{\nu\nu}$$

$$\|$$

$$\sum_{g\in G} \chi_{\rho_2}(g^{-1}) \chi_{\rho_1}(g) = \sum_{g\in G} \chi^*_{\rho_2}(g) \chi_{\rho_1}(g) = \begin{cases} 0 & \text{if } \rho_1 \not\cong \rho_2 \\ |G| & \text{if } \rho_1 \cong \rho_2 \end{cases}$$

$$\Rightarrow \frac{1}{|G|}\sum_{g\in G} \chi^*_{\rho_2}(g)\chi_{\rho_1}(g) = \begin{cases} 0 & \text{if } \rho_1 \not\cong \rho_2 \\ 1 & \text{if } \rho_1 \cong \rho_2 \end{cases}$$

$$\overbrace{\langle \chi_{\rho_2}, \chi_{\rho_1} \rangle}^{\text{|||}}$$

Characters for irreducible representations are orthonormal under this inner product!

$\Rightarrow$ if we have a reducible representation

$$\rho \cong \underbrace{\rho_1 \oplus \rho_1 \oplus \cdots \oplus \rho_1}_{n_1} \oplus \underbrace{\rho_2 \oplus \rho_2 \oplus \cdots \rho_2}_{n_2} \oplus \cdots$$

$$\chi_\rho = n_1 \chi_{\rho_1} + n_2 \chi_{\rho_2} + \dots = \sum_{i=1}^{N_{irr}} n_i \chi_{\rho_i} \leftarrow \text{irreducible}$$

$$\langle \chi_{\rho_j}, \chi_\rho \rangle = n_j$$