Lecture 4 | Recap: representation ρ of a group G

- a vector space V
- a group homomorphism $\rho: G \to U(V)$ to unitary operators on V

Let G be a group $\rho: G \to U(V)$ a representation on V

Consider a vector $|v\rangle \in V$

$g \in G$ $\qquad \rho(g) \in U(V)$

$\rho(g)|v\rangle$ is also in V

look for subspaces $W \subseteq V$ such that
$\rho(g)|w\rangle \in W$ for $|w\rangle$ in W and all $g \in G$

such a subspace is called an <u>invariant subspace</u>

Given an invariant subspace, we can look at $W^{\perp}$ - orthogonal complement

$$W^{\perp} = \{ |w_{\perp}\rangle \in V \mid \langle w_{\perp}|w\rangle = 0 \text{ for all } |w\rangle \in W \}$$

$V = W \oplus W^{\perp}$ $\leftarrow$ every $|v\rangle \in V$ can be written as

$$|v\rangle = \underset{\substack{\in \\ W}}{|w\rangle} + \underset{\substack{\in \\ W^\perp}}{|w_\perp\rangle}$$

It turns out $W^\perp$ is <u>also</u> an invariant subspace

pf: take any $|w\rangle \in W$ and $|w_\perp\rangle \in W^\perp$

for all $g \in G$ $\quad \rho(g)|w\rangle \in W$

$$\Rightarrow \left(\langle w_\perp|\rho(g)|w\rangle\right)^* = (0)^*$$

$$\langle w|\rho^\dagger(g)|w_\perp\rangle = 0$$

$$\langle w|\rho(g^{-1})|w_\perp\rangle = 0$$

$\Rightarrow \rho(g)|w_\perp\rangle \in W^\perp$ for any $g \in G$

$\Rightarrow W^\perp$ is an invariant subspace too

Practically speaking: basis for $V = \{|v_1\rangle, |v_2\rangle, |v_3\rangle, \ldots\}$

$$\rho(g) = \sum_{i,j} |v_i\rangle \underbrace{\langle v_i|\rho(g)|v_j\rangle}_{\rho_{ij}(g)} \langle v_j| \quad \text{"}A\text{"}$$

$\rho_{ij}(g)$ is a matrix acting on column vectors in this basis

$$\rho_{ij}(g) = \begin{pmatrix} \langle v_1|\rho(g)|v_1\rangle & \langle v_1|\rho(g)|v_2\rangle & \cdots \\ \langle v_2|\rho(g)|v_1\rangle & \langle v_2|\rho(g)|v_2\rangle & \\ \vdots & & \ddots \end{pmatrix}$$

Using $V = W \oplus W^{\perp}$ we can pick a new basis

$$B = \{ \underbrace{|w_1\rangle, |w_2\rangle, \ldots}_{\text{basis for } W}, \underbrace{|w_1^{\perp}\rangle, |w_2^{\perp}\rangle, \ldots}_{\text{basis for } W^{\perp}} \}$$

$_W B = A\, U_{W^{\perp}}$ some unitary matrix

$$[\rho(s)] = \left(\begin{array}{c|c} \langle w_i|\rho(s)|w_j\rangle & \langle w_i|\rho(s)|w_j^{\perp}\rangle \\ \hline \langle w_i^{\perp}|\rho(s)|w_j\rangle & \langle w_i^{\perp}|\rho(s)|w_j^{\perp}\rangle \end{array}\right) = \begin{array}{c} W \\ W^{\perp} \end{array} \left(\begin{array}{c|c} \rho_W(s) & 0 \\ \hline 0 & \rho_{W^{\perp}}(s) \end{array}\right)$$

$\Rightarrow$ one choice of basis block-diagonalises every $\rho(g)$ for all $g \in G$

$$\rho_W : G \to U(W)$$

$$\rho_{W^\perp} : G \to U(W^\perp)$$

$\rho \cong \rho_W \oplus \rho_{W^\perp}$ are equivalent representations

subrepresentations

$\Rightarrow$ ρ is a reducible representation if and only if it has a nontrivial invariant subspace

(not V or $\{\vec{0}\}$)

if a representation is not reducible we say its irreducible

Trivial example: let G be any group $V = \mathbb{C}$ complex numbers

$$U(\mathbb{C}) = \{e^{i\phi} \mid \phi \in [0, 2\pi)\} = U(1)$$

take $\rho: G \to U(1)$ $\rho(g) = 1 = e^{i0}$

$$\rho(g_1 g_2) = 1 = \rho(g_1)\rho(g_2) = (1)(1)$$

- trivial representation - irreducible

Nontrivial: SU(2)

spin-$\frac{1}{2}$ representation

$$V_{\frac{1}{2}} = \{ |\uparrow\rangle, |\downarrow\rangle \}$$

$$\rho_{\frac{1}{2}} : (\hat{n}, \theta) \to e^{-i\frac{\theta}{2}\vec{\sigma}\cdot\hat{n}}$$

Consider now the space of <u>two</u> spin-$\frac{1}{2}$ particles

$$V = V_{\frac{1}{2}} \otimes V_{\frac{1}{2}} = \{ |\uparrow\uparrow\rangle, |\downarrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\downarrow\rangle \}$$

$$\rho_{\frac{1}{2}\times\frac{1}{2}}(\hat{n}, \theta) \longrightarrow e^{-i\frac{\theta}{2}\vec{\sigma}_1\cdot\hat{n}} \otimes e^{-i\frac{\theta}{2}\vec{\sigma}_2\cdot\hat{n}}$$

Are there nontrivial invariant subspaces for $\rho_{\frac{1}{2}\otimes\frac{1}{2}}$?

$$V_0 = \left\{ \frac{1}{\sqrt{2}} \left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \right) \right\} \quad \text{- singlet state}$$

$J=0$

orthogonal complement

$$\begin{array}{cccc} & m=1 & m=0 & m=-1 \\ V_1 = \Big\{ & |\uparrow\uparrow\rangle, & \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right), & |\downarrow\downarrow\rangle \Big\} \end{array}$$

$J=1$ triplet

$$\rho_{\frac{1}{2}\otimes\frac{1}{2}}(\hat{n},\theta) = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & e^{-i\hat{n}\cdot\vec{L}\theta} \end{array} \right)$$

L - spin-1 matrices

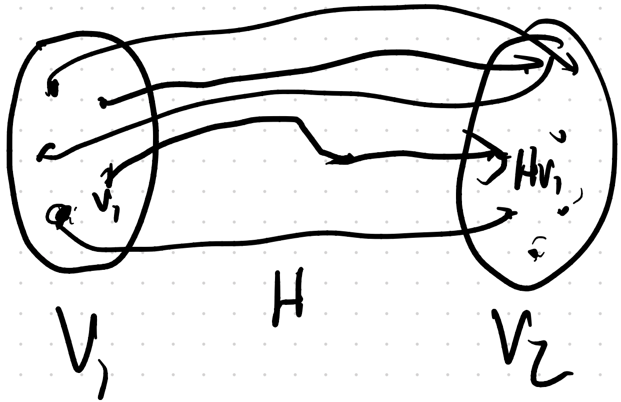
$$\rho_{\frac{1}{2}\times\frac{1}{2}} \simeq \rho_0 \oplus \rho_1 \qquad \left(\tfrac{1}{2}\otimes\tfrac{1}{2} = 0 \oplus 1\right)$$

Clebsch Gordan coeffs: matrix elements of the change of basis that block-diagonalize ρ

Schur's Lemma (2 $\frac{1}{2}$ parts)

Schurs lemma part 1 Let G be a group

$\rho_1 : G \to U(V_1)$
$\rho_2 : G \to U(V_2)$
two irreducible representations



(irreps)

and let $H: V_1 \to V_2$ be a linear map (rectangular matrix

Suppose $H\rho_1(g) = \rho_2(g)H$ for all $g \in G$

(intertwining reln)

Then: either ① $H = 0$

or ② H is invertible

Pf: $\boxed{\begin{array}{ll} \text{if } H=0 & \ker H = V_1 \\ & \operatorname{Im} H = \{\vec{0}\} \\ \text{if } H \text{ is invertible} & \operatorname{Im} H = V_2 \\ & \ker H = \{\emptyset\} \end{array}}$

$$H\rho_1(g) = \rho_2(g)H$$

look at $\ker H = \{|v\rangle \in V_1 \mid H|v\rangle = 0\}$

$|v_1\rangle \in \ker H$

$$H\rho_1(g)|v_1\rangle = \rho_2(g)H|v_1\rangle = 0$$

$\rightarrow \ker H$ is an invariant subspace of ρ_1

But ρ_1 is irreducible

$$\rightarrow \ker H = \begin{cases} \{\vec{0}\} \\ V_1 \end{cases}$$

$$\mathrm{Im}\, H = \{ |v_2\rangle \in V_2 \mid |v_2\rangle = H|v_1\rangle \text{ for } |v_1\rangle \in V_1 \}$$

$|w\rangle \in \mathrm{Im}\, H$ $\qquad |w\rangle = H|v\rangle \qquad |v\rangle \in V_1$

$$\underset{\underset{\mathrm{Im}\, H}{\in}}{\rho_2(g)|w\rangle} = \rho_2(g) H|v\rangle = H \rho_1(g)|v\rangle$$

$\mathrm{Im}\, H$ is an invariant subspace of

ρ_2 $\quad$ H invertible

$$\operatorname{Im} H = \begin{cases} V_2 \\ \vec{0} \end{cases} \qquad \ker H = \begin{cases} \{\vec{0}\} \\ V_1 \end{cases}$$

$$H = 0$$

Part 2 $\quad V_1 = V_2 = V$ and $\rho_1 = \rho_2 = \rho$

the same vector space, and the same irreducible representation and V finite-dimensional

and suppose H satisfies the intertwining relation

$H\rho(g) = \rho(g) H$ for all $g \in G$

$\Downarrow$

$[H, \rho(g)]$

then: ① $H = 0$

② $H = \lambda \, Id_V$ identity operator

Proof: Part 1 says $H = 0$ or H is invertible

so assume H invertible

$\Rightarrow$ H is an invertible finite-dimensional square matrix

$\Rightarrow$ H has at least one eigenvector $|v\rangle$
with eigenvalue λ

$B = H - \lambda \, \mathrm{Id}_V \rightarrow [B, \mathrm{ers}] = 0$

$\downarrow$

Part 1 says $B = 0$ or B is invertible

but $\det B = 0 \rightarrow B$ can't be invertible

$\Rightarrow B = 0 = H - \lambda \, \mathrm{Id}_V$

$\Rightarrow H = \lambda \, \mathrm{Id}_V$

Part 2.5

G a group

$\rho_1 : G \to U(V_1)$

$\rho_2 : G \to U(V_2)$

finite dimensional irreducible representations

$H : V_1 \to V_2$

Suppose $H\rho_1(g) = \rho_2(g)H$ for all $g \in G$ and $H \neq 0$

Part 1: H is invertible

Then $\rho_1 \cong \rho_2$

Proof: $H^{\dagger} : V_2 \to V_1$ is a matrix from $V_2 \to V$

$$\left(H\rho_1(g^{-1}) = \rho_2(g^{-1})H\right)^{+}$$

$$\rho_1(g)H^{+} = H^{+}\rho_2(g)$$ H^{+} also satisfies the assumptions of part 1

$\Rightarrow H^{+}$ is invertible

$H^{+}H : V_1 \to V_2 \to V_1$ and $[H^{+}H, \rho_1(g)]$

$\Rightarrow$ part 2 $H^{+}H = \lambda \, Id_{V_1}$

$$H^{+} = \lambda H^{-1}$$

define $U = \frac{1}{\sqrt{\lambda}} H$

$$U^{\dagger} = \frac{1}{\sqrt{2}} H^{\dagger} = \sqrt{2}\, H^{-1} = U^{-1}$$

$$U \rho_1(g) = \rho_2(g)\, U$$

$$U \rho_1(g) U^{\dagger} = \rho_2(g) \quad \to \quad \rho_1(g) \simeq \rho_2(g)$$