Lecture 3

Announcements: - HW1 will be posted today
Due 2/13 11:59 PM central
Thursday

- Office hours start Next week
  - My office hours: 2-3 Tuesdays
    via Zoom (link will be posted on course website)

Recap: Groups, subgroups, cosets → First Isomorphism theorem

let G, K be groups and

$\phi: G \to K$ a homomorphism $[\phi(g_1 \cdot g_2) = \phi(g_1) \cdot \phi(g_2)]$

$\{\varphi(g) \mid g \in G\} = \operatorname{Im}\varphi \leq K$

$\{g \mid \varphi(g) = E_K\} = \operatorname{Ker}\varphi \trianglelefteq G$

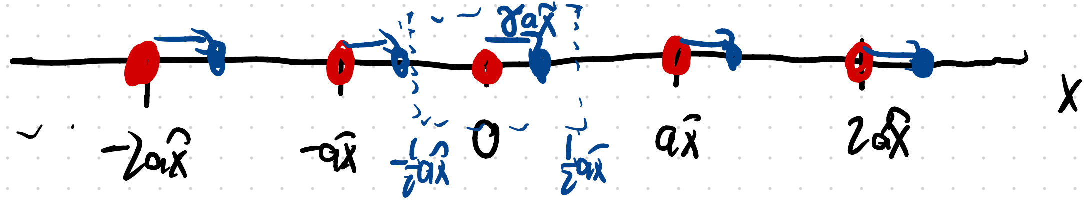
Then $\operatorname{Im}\varphi \cong G/\operatorname{Ker}\varphi$

isomorphic: theres an invertible homomorphism

$\psi: G/\operatorname{Ker}\varphi \to \operatorname{Im}\varphi$

Example: translations in 1D $\mathbb{R} = \{\beta a\hat{x} \mid \beta \text{ is real}\}$

addition of vectors



Subgroup: $T = \{n a\hat{x} \mid n \in \mathbb{Z}\}$

Coset decomposition: $\mathbb{R} = \bigcup_{\gamma \in [-\frac{1}{2}, \frac{1}{2}]} \left(T + \gamma a \hat{x} = \{ n a\hat{x} + \gamma a \hat{x} \mid n \in \mathbb{Z}, \gamma \in \mathbb{R} \} \right)$

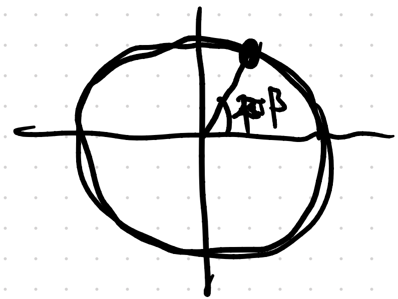
Claim: $\mathbb{R}/T \cong U(1)$ the unit circle

Pf by 1st isomorphism theorem

$$\varphi : \mathbb{R} \to U(1) \quad \text{s.t.} \quad \operatorname{Ker}\varphi = T$$

$$\operatorname{Im}\varphi = U(1)$$

$$\varphi(\beta a\hat{x}) = e^{2\pi i\beta}$$



$$\operatorname{Im}\varphi = U(1)$$

$$\varphi(\beta_1 a\hat{x} + \beta_2 a\hat{x})$$

$$\|$$

$$e^{i2\pi(\beta_1+\beta_2)} = e^{2\pi i\beta_1} e^{2\pi i\beta_2} = \varphi(\beta_1 a\hat{x})\,\varphi(\beta_2 a\hat{x}) \checkmark$$

$$\operatorname{Ker}\varphi = \left\{ \beta a\hat{x} \,\middle|\, e^{2\pi i\beta} = 1 \right\} = T$$

$\Rightarrow \quad \mathbb{R}/T = U(1)$ - unit cell of the Bravais lattice

One last point about quotient groups:

let $H \triangleleft G$ be a normal subgroup

$$G = \bigcup_{i=0}^{n-1} Hg_n \qquad g_0 = E$$

$G/H = \{H, Hg_1, Hg_2, \dots Hg_{n-1}\}$ is a group

In <u>some cases</u> we can relate elements of G/H to elements of G

sometimes there exists a homomorphism

$$i : G/H \to G$$

$$i(Hg_i) = \tilde{g}_i \in Hg_i$$

$i(H) = E$

If i exists, then $\mathrm{Im}\, i = \{E, \tilde{g}_1, \tilde{g}_2, \ldots \tilde{g}_{n-1}\} \leq G$

and from 1st Isomorphism Thm, $\mathrm{Im}\, i \cong G/H$

and $G = \bigcup_{i=0}^{n-1} H(\mathrm{Im}\, i) \leftarrow$ every element of G

can written uniquely as $g = hk$ $\quad h \in H$, $k \in \mathrm{Im}\, i$

if this is possible, we say

G is a <u>semidirect product</u> $H \rtimes (\mathrm{Im}\, i)$ $\quad H \triangleleft G$, $\mathrm{Im}\, i < G$

Example: Rigid transformations of 3D space

Euclidean group $\mathbb{E}(3)$
- translations
- rotations
- reflections

$\{R|\vec{v}\} = g \in \mathbb{E}(3)$

"Seitz symbol" for g

$R \in O(3)$ rotation or reflection (orthogonal transformation)

$\vec{v} \in \mathbb{R}^3$ translation

$[R]$ - 3x3 matrix that implements R

Action on points in space: $\{R|\vec{v}\}\vec{x} = [R]\vec{x} + \vec{v}$

multiplication in $\mathbb{E}(3)$:

$$g_1 = \{R_1 | \vec{v}_1\}$$
$$g_2 = \{R_2 | \vec{v}_2\}$$

$$g_1 \cdot g_2 \vec{x} = g_1 \cdot (g_2 \vec{x})$$
$$= g_1 ([R_2]\vec{x} + \vec{v}_2)$$
$$= [R_1]([R_2]\vec{x} + \vec{v}_2) + \vec{v}_1$$
$$= [R_1 R_2]\vec{x} + (\vec{v}_1 + [R_1]\vec{v}_2)$$
$$= \{R_1 R_2 | \vec{v}_1 + R_1 \vec{v}_2\} \vec{x}$$

multiplication rule: $\{R_1 | \vec{v}_1\}\{R_2 | \vec{v}_2\} = \{R_1 R_2 | \vec{v}_1 + R_1 \vec{v}_2\}$

inverses $\{R|\vec{v}\}^{-1} = \{R^{-1}| - R^{-1}\vec{v}\}$

identity: $\{E|\vec{0}\}$

① $O(3) < \mathbb{E}(3)$
$\quad\ \ \cong$
$\{\{R|\vec{0}\} \mid R \in O(3)\}$

② $\mathbb{R}^3 = \{\{E|\vec{v}\} \mid \vec{v} \in \mathbb{R}^3\} \triangleleft \mathbb{E}(3)$ is a normal subgroup

$\{R|\vec{d}\}\{E|\vec{v}\}\{R|\vec{d}\}^{-1} = (\{R|\vec{d}\}\{E|\vec{v}\})\{R^{-1}|-R^{-1}\vec{d}\}$
$= \{R|\vec{d}+R\vec{v}\}\{R^{-1}|-R^{-1}\vec{d}\}$

$$= \{E \mid R\vec{v}\} \in \mathbb{R}^3$$

Finally $\{R \mid \vec{v}\} = \{E \mid \vec{v}\}\{R \mid \vec{0}\}$

$$\mathbb{E}(3) = \mathbb{R}^3 \rtimes O(3)$$

Right coset decomposition $\mathbb{E}(3) = \bigcup_{R \in O(3)} \mathbb{R}^3 R$

How we use groups in Condensed Matter physics

$H = \frac{p^2}{2m} + V(\vec{x}) + \ldots$ group of symmetries $g \in G$

$$\vec{x} \to \vec{x}' = g\vec{x}$$

$$\vec{p} \to \vec{p}' = g\vec{p}$$

$$\psi'(\vec{x}) \to \psi(g^{-1}x)$$

$$H \to H' = H$$

V - QM Hilbert space

$|\psi\rangle \in V$

$G \ni g \to \rho(g) \in U(V)$ - group of unitary operators on V

$\rho(g)|\psi\rangle = |\psi'\rangle$

$\rho^\dagger(g)\,\vec{x}\,\rho(g) = x'$

$\rho^\dagger(g)\,\vec{p}\,\rho(g) = p'$

$\rho: G \to U(V)$ should be a homomorphism

$\rho(g_1 g_2) = \rho(g_1)\rho(g_2)$

Define: a (unitary) <u>representation</u> of a group G is:

a vector space V

a homomorphism $\rho: G \to U(V)$

unitary matrices/operators on V

V is called the representation space

$\rho(g)$ is called the representative of G

Example: representation of translations $\mathbb{R}^3$ on the hilbert space $V = \{ \psi(\vec{x})$ square-integrable wave functions $\}$

$$\mathbb{R}^3 \ni \vec{V} \rightarrow \rho(\vec{V}) = e^{-\frac{i}{\hbar}\vec{p}\cdot\vec{V}}$$

homomorphism $\rho(\vec{V}_1 + \vec{V}_2) = e^{-\frac{i}{\hbar}\vec{p}\cdot(V_1+V_2)} = e^{-\frac{i}{\hbar}\vec{p}\cdot V_1} e^{-\frac{i}{\hbar}\vec{p}\cdot V_2} = \rho(V_1)\rho(V_2)$

$$\rho(\vec{V})^\dagger \vec{x} \rho(\vec{V}) = e^{+\frac{i}{\hbar}\vec{p}\cdot\vec{V}} \vec{x} e^{-\frac{i}{\hbar}\vec{p}\cdot\vec{V}}$$

$$= \hat{x} + \frac{i}{\hbar}[\vec{V}\cdot\vec{p}, \vec{x}] = \vec{X} + \vec{V}$$

More complicated example: $SU(2) \leftarrow$ special unitary group 2D

$$SU(2) = \{ (\hat{n}, \theta) \mid \theta \in (-2\pi, 2\pi) \}$$

3D unit vector

$$(\hat{n}, \theta = 0) = E$$
$$(\hat{n}, \theta = 2\pi) = \bar{E}$$

Spin-$\frac{1}{2}$ representation of $SU(2)$

$$\rho_{\frac{1}{2}}((\hat{n}, \theta)) = \cos\frac{\theta}{2}\,\sigma_0 + i\sin\frac{\theta}{2}\,\hat{n}\cdot\vec{\sigma}$$

2x2 identity matrix

vector of Pauli matrices

$$\| \quad e^{-i\frac{\theta}{2}\hat{n}\cdot\vec{\sigma}}$$

<u>defining representation</u>

$$\operatorname{Ker}\rho_{\frac{1}{2}} = \{E\}$$
$$\operatorname{Im}\rho_{\frac{1}{2}} = SU(2)$$

Other representations: Spin-$l=1$

$$\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} = L'_x \leftarrow L_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} = L'_y \leftarrow L_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = L'_z \leftarrow L_z = \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}$$

$$\rho_1((\hat{n},\theta)) = e^{-i\hat{n}\cdot\vec{L}\theta}$$

$$\mathrm{Ker}\,\rho_1 = \{E, \bar{E}\}$$

$$\mathrm{Im}\,\rho_1 \cong SU(2)/\{E,\bar{E}\}$$

$$\mathrm{Im}\,\rho_1 \cong SO(3) \ni e^{-i\hat{n}\cdot\vec{L}'\theta}$$

"SU(2) is a double cover of SO(3)"

Two representations $\rho : G \to U(V)$

$\sigma : G \to U(V)$

ρ and σ are <u>equivalent representations</u> $\rho \cong \sigma$

if there is a unitary matrix A

$$A\rho(g)A^{\dagger} = \sigma(g) \text{ for all } g \in G$$

i.e. if the only difference is a change of basis

A

$$r^2 = \frac{E}{E}$$