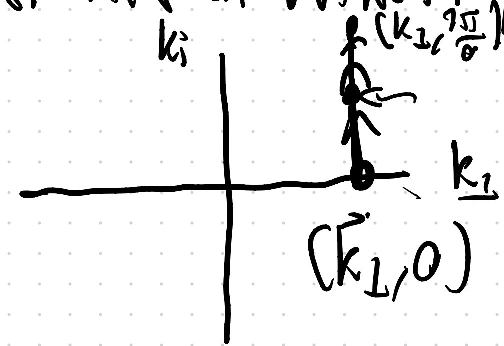
# Lecture 22

Reminders:
- HW 4 due tonight
- HW due 5/7 at 11:59pm

Last time: Inversion symmetry constraints on Wilson loops

for $k_\perp = -k_\perp + \vec{b}_\perp \quad \vec{b}_\perp \in \Gamma^\vee$




$$W_{\frac{2\pi}{a}\leftarrow 0}(k_\perp) = B_{(k_\perp,0)}(I)\, W^\dagger_{\frac{\pi}{a}\leftarrow 0}(k_\perp)\, B_{(k_\perp,\frac{\pi}{a})}(I)\, W_{\frac{\pi}{a}\leftarrow 0}(k_\perp)$$

$$B^{nm}_{\vec{k}}(I) = u^\dagger_{nk} \cdot V(Ik-k) B(I) \vec{u}_{mk} \quad \text{for } k \text{ a TRIM}$$

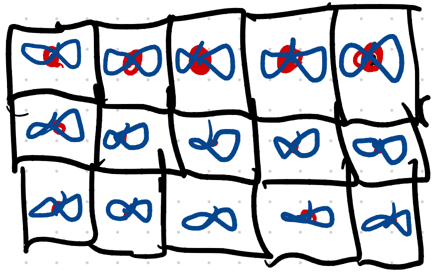
little group representation matrices for $I$

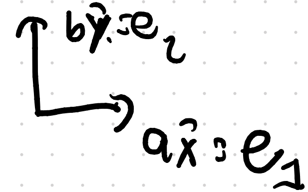
$$\det W_{\frac{2\pi}{a}\leftarrow 0}(k_\perp) = \det B_{(k_\perp,0)}(I)\, \det B_{(k_\perp,\frac{\pi}{a})}$$

$$e^{i\sum_{l=1}^{N_{occ}} \varphi_l(k_\perp)} = e^{i\pi\left(n_-^{(k_\perp,0)} + n_-^{(k_\perp,\frac{\pi}{a})}\right)}$$

$n_-^{k}$ # of negative eigenvalues of $B_k(I)$

Lets extend this to 2D






two-band Nearest-Neighbor
tb model

1D: $h(k_x) = (\Delta + t_1 \cos k_x a)\sigma_z + t_2 \sin k_x \sigma_y$

$$\downarrow$$

$h(k_x, k_y) = (\Delta + t_1 \cos k_x a)\sigma_z + t_2 \sin k_x \sigma_y$

This is boring

Two phases of $h(k_x)$

1. $\Delta > t_1 > 0$ $\quad \varphi_- = 0$

$$② \quad t_1 > \Delta > 0 \quad \varphi_- = \pi$$
$$t_2 \neq 0$$

$h(k_x, k_y = 0)$ - 1D hamiltonian in phase ①

$h(k_x, k_y = \frac{\pi}{b})$ - 1D hamiltonian in phase ②

$$\Delta \to \Delta(1 + \cos k_y b)$$

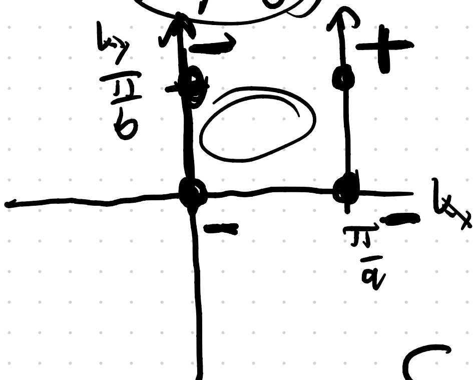
$$\Delta > t_1 > 0$$

$$h(k_x, k_y) = (\Delta + \Delta \cos k_y b + t_1 \cos k_x a)\sigma_z + t_2 \sin k_y a \, \sigma_y$$

$$h(k_x, k_y = 0) = [2\Delta + t_1 \cos k_x a]\sigma_z + t_2 \sin k_y a \, \sigma_y$$

phase (1)

$$h(k_x, k_y=\tfrac{\pi}{b}) = t_1 \cos k_x a\, \sigma_y + t_2 \sin k_x a\, \sigma_x \quad \leftarrow \text{phase (2)}$$




Energies:

$$\varepsilon_\pm = \pm\sqrt{(\Delta+\Delta\cos k_y b + t_1\cos k_x a)^2 + t_2^2 \sin^2 k_x a}$$

$$k_x = \frac{\pi}{a}$$

$$\varepsilon_\pm(k_x=\tfrac{\pi}{a}, k_y) = \pm(\Delta+\Delta\cos k_y b - t_1)$$

$$\cos k_y b = \frac{t_1}{\Delta} - 1 \quad \text{gapless point}$$

Need to fix this → add an extra term to $h$

$$+ f(k)\sigma_x$$

$$\rightarrow \varepsilon_{\pm}(\vec{k}) = \pm\sqrt{(\Delta + \Delta\cos k_y b + t_1 \cos k_x a)^2 + t_2^2 \sin^2 k_x a + f(k)^2}$$

$$B(I) = \sigma_z$$

$$B(T) = \sigma_0 K$$

Inversion symmetry

$$\sigma_z f(k)\sigma_x \sigma_z = f(-k)\sigma_x$$

$$\Rightarrow f(k) = -f(-\vec{k})$$

Give up TRS

$$f(k) = -t_2 \sin k_y b$$

TRS:

$$\cancel{f^*(k) = f(k) = f(-k)}$$

$$h^{2D}(k_x, k_y) = \left[\Delta(1+\cos k_y b) + t_1 \cos k_x a\right]\sigma_z + t_2 \sin k_x a\,\sigma_y - t_2 \sin k_y b\,\sigma_x$$

Gapped spectrum $\forall k$ when $\Delta > t_1 > 0$, $t_2 \neq 0$

Wilson loop for negative energy band as a function of $k_x$

$$W_{\frac{2\pi}{b}\leftarrow 0}(k_x) = \vec{u}^{\dagger}_{-,(k_x,0)} \cdot V\left(\frac{2\pi}{b}\hat{y}\right) \cdot \prod_{k_y}^{\frac{2\pi}{b}\leftarrow 0} \hat{P}(\vec{k}) \cdot \vec{u}_{-,(k_x,0)}$$

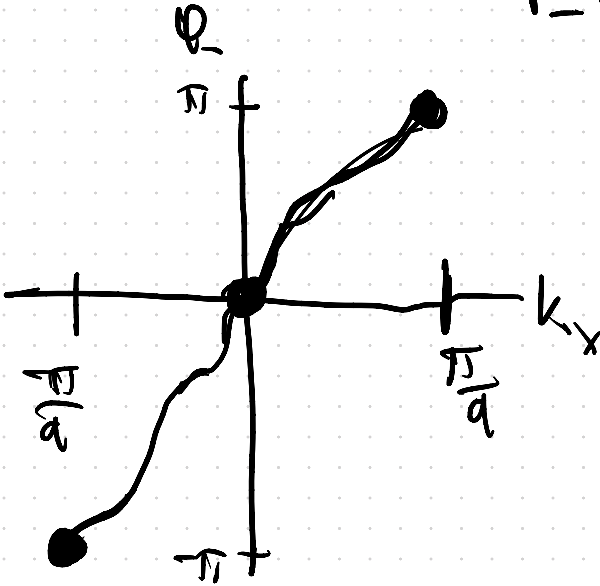
$$\det W_{\frac{2\pi}{b}\leftarrow 0}(k_x) = e^{i\varphi_-(k_x)} = W_{\frac{2\pi}{b}\leftarrow 0}(k_x)$$

at $k_x = 0$

$$\varphi_-(k_x=0) = 0 \mod 2\pi$$

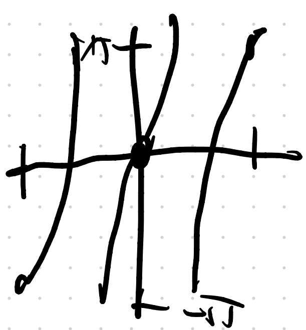
at $k_x = \frac{\pi}{a}$

$$\varphi_-\left(k_x=\frac{\pi}{a}\right) = \pi \mod 2\pi$$




$\varphi_-(k_x)$ is related to the eigenvalues of $PyP$

$$PyP|W_{-,k_x,R_y}\rangle = \left(R_y + \frac{b}{2\pi}\varphi(k_x)\right) \times |W_{-,k_x,R_y}\rangle$$




$$I\,PyP\,I^{-1} = -PyP$$

Inversion: $\varphi_-(k_x) = -\varphi_-(-k_x)$

Define: $C \in \mathbb{Z}$ - the # of times $\varphi(k_x)$ winds from $-\pi$ to $\pi$ as $k_x$ goes from $-\frac{\pi}{a} \to \frac{\pi}{a}$

C <u>must</u> be odd for our model

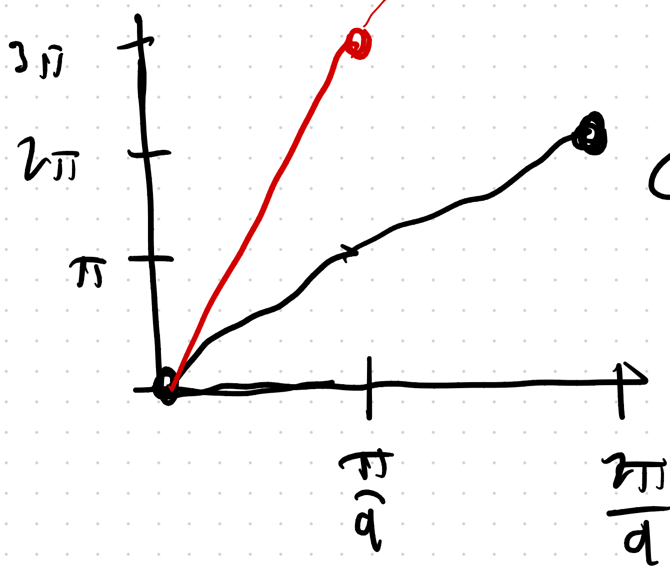
Formula for C

$$e^{i\varphi_-(k_x)} = \det W(k_x) = \det W\left(k_x + \frac{2\pi}{a}\right) = e^{i\varphi_-\left(k_x + \frac{2\pi}{a}\right)}$$

$6\pi$

$$\varphi_-\left(k_x + \frac{2\pi}{a}\right) = \varphi_-(k_x) + 2\pi C$$

$4\pi$




$C = 1$

$$\varphi_-(k_x) = Cak_x + g(k_x)$$

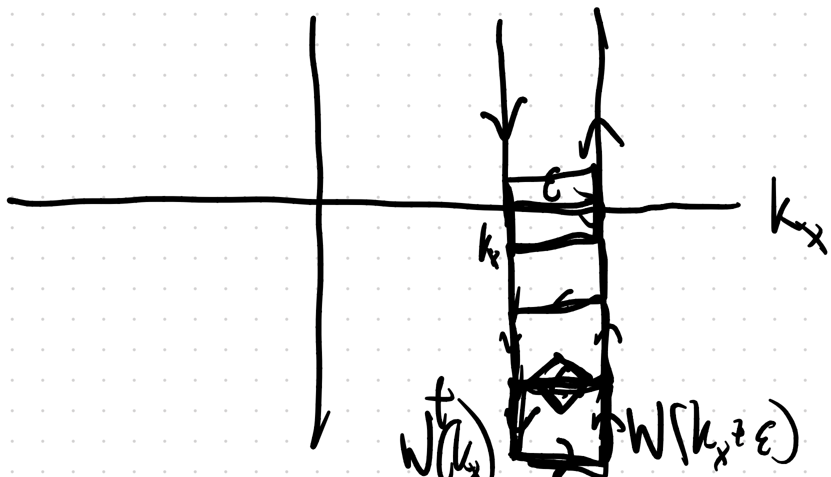
where $g(k_x + \frac{2\pi}{a}) = g(k_x)$ is periodic

$$\frac{1}{2\pi}\int_{-\pi/a}^{\pi/a} dk \frac{\partial \varphi_-}{\partial k_x} = \frac{1}{2\pi}\int_{-\pi/a}^{\pi/a} dk_x \left(Ca + \cancel{g'(k_x)}^{\,0}\right) = C$$

$$\boxed{C = \frac{1}{2\pi}\int_{-\pi/a}^{\pi/a} dk_x \frac{\partial}{\partial k_x} \operatorname{Im} \log \det W(k_x)}$$

$$\frac{\partial}{\partial k_x} \operatorname{Im} \log\det W = \operatorname{Im} \lim_{\varepsilon \to 0} \frac{\log\det W(k_x+\varepsilon) - \log\det W(k_x)}{\varepsilon}$$

$$= \operatorname{Im} \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \log\det\left[W(k_x+\varepsilon) W^{\dagger}(k_x)\right]$$



$$\frac{\partial}{\partial x} \operatorname{Im} \log\det W = \operatorname{Im} \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \sum_{\text{squares}} \log\det W_{\square}$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \sum_{\text{squares}} \oint_{\text{square}} dk \operatorname{tr} A(\vec{k})$$

$$\Omega_{xy} = \frac{\partial A_y}{\partial k_x} - \frac{\partial A_x}{\partial k_y} - i[A_x, A_y]$$

$$= \int_{-\pi/b}^{\pi/b} dk_y \operatorname{tr} \Omega_{xy}(k_x, k_y)$$

$$C = \frac{1}{2\pi} \int_{-\pi/a}^{\pi/a} dk_x \int_{-\pi/b}^{\pi/b} dk_y \operatorname{tr} \Omega_{xy}(k_x, k_y)$$

$$\boxed{C = \frac{1}{2\pi} \oiint d^2k \operatorname{tr} \Omega_{xy}(k_y, k_z)}$$

BZ

Chern number $\in \mathbb{Z}$

Physical meaning of $C$: Consider Hybrid WFs

$$|W_{-,k_x,R_y}\rangle$$

$$P_y P |W_{-,k_x,R_y}\rangle = \left[R_y + \frac{b}{2\pi}\varphi_-(k_x)\right]|W_{-,k_x,R_y}\rangle$$

$$P_y P |W_{-,k_x+\frac{2\pi}{a},R_y}\rangle = \left[R_y + \frac{b}{2\pi}\varphi_-\left(k_x+\frac{2\pi}{a}\right)\right]|W_{-,k_x+\frac{2\pi}{a},R_y}\rangle$$

$$= \left[ R_y + bC + \frac{b}{2\pi} \varphi_-(k_x) \right] \left| W_{-, k_x + \frac{2\pi}{a}, R_y} \right\rangle$$

$$\left| W_{-, k_x + \frac{2\pi}{a}, R_y} \right\rangle = \left| W_{-, k_x, R_y + Cb} \right\rangle$$

Apply an electric field $\vec{E} = E_0 \hat{x}$

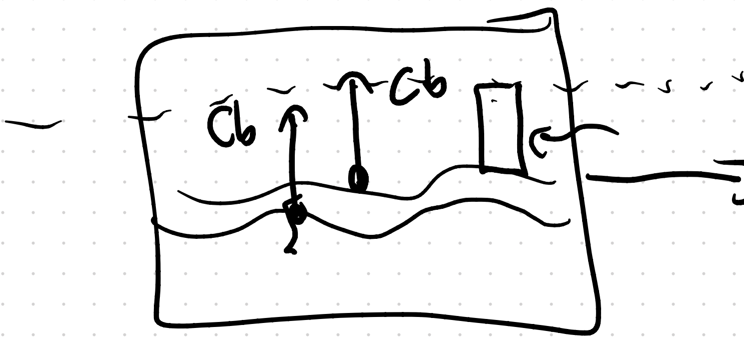
$$\vec{A} = -E_0 t \hat{x} \qquad \vec{E} = -\frac{\partial A}{\partial t}$$

$$\vec{p} \to \vec{p} - q\vec{A} = \vec{p} + qE_0 t \hat{x}$$

$$k \to k(t) = (k_x + qE_0 t, k_y)$$

$$|W_{-,k_x,R_y}\rangle \to |W_{-,k_x+qE_0t,R_y}\rangle$$

after one period $\frac{qE_0T}{\hbar} = \frac{2\pi}{a} \to T = \frac{2\pi\hbar}{aqE_0}$




$\to E_x \quad \Delta Q = N_x C q$

$$I_y = \frac{\Delta Q}{T} = Cq\left(\frac{aqE_0}{2\pi\hbar}\right)N_x$$

$$= C\frac{q^2}{h}V_x$$

$N_x a E_0 = V_x$

Voltage drop

$$G_H = \text{Integer} \times \frac{e^2}{h}$$

→ The Chern number is proportional to the Hall conductance