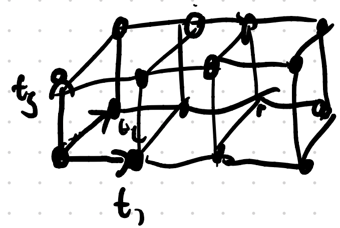
Lecture 2

Recap: group G - set w/ associative "multiplication," identity element, inverses

Ex: Bravais lattice $T = \{ \sum_{i=1}^{3} n_i \vec{t}_i \mid t_i$ linearly independent $n_i \in \mathbb{Z} \}$



- binary operation: $+$
- $\vec{0} \in T$ is the identity
- $\left(\sum_i n_i \vec{t}_i\right)^{-1} = \sum_i (-n_i) \vec{t}_i$

Subgroups: $H \leq G$ if $H \subseteq G$ and H is also a group

(Note: Every group has at least 2 subgroups

$G \leq G$ - the whole group

$\{E\} < G$ "trivial subgroup")

right cosets $Hg_i = \{hg_i \mid h \in H\}$

partition the group $G = \bigcup_{i=0}^{n-1} Hg_i$ $n = |G:H|$ index of H in G

Define: conjugation by $g_i \in G$

$$C_{g_i}(g) : g \to g_i \cdot g \cdot g_i^{-1}$$

We say two group elements $g_1, g_2 \in G$ are <u>conjugate</u> if there's some $g_a \in G$ s.t.

$$C_{g_a}(g_1) = g_a g_1 g_a^{-1} = g_2$$

$$C_{g_a^{-1}}(g_2) = g_a^{-1} g_2 g_a = g_1$$

$$\mathcal{C}(g) = \{ g_a g g_a^{-1} \mid g_a \in G \}$$ - conjugacy class of

$g = \{\text{all elements conjugate to } g\}$

If $a = g_1 b g_1^{-1}$ and $b = g_2 c g_2^{-1}$

$\Rightarrow a = g_1 (g_2 c g_2^{-1}) g_1^{-1}$

$= (g_1 g_2) c (g_1 g_2)^{-1}$

a conjugate to b and b conjugate to $c \Leftrightarrow a$ conjugate to c

Given a subgroup $H \leq G$ and $g \in G$

$g \cdot H g^{-1} = \{ g h g^{-1} \mid h \in H \}$

$gHg^{-1} \leq G$ conjugate subgroup

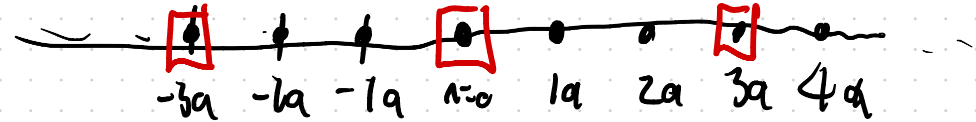
Def $H \leq G$ is a normal subgroup if $gHg^{-1} = H$ for any $g \in G$

$$Hg = gH$$

we write this as $H \trianglelefteq G$

$(H \neq G$ and H normal $H \triangleleft G)$

Example: 1D Bravais lattice $T = \{ n a \hat{x} \mid n \in \mathbb{Z} \}$ a ~ lattice constant



$$H = \{3na\hat{x} \mid n \in \mathbb{Z}\}$$

$$H = \{0, 3a\hat{x}, -3a\hat{x}, \dots\}$$

$$H + a\hat{x} = \{a\hat{x}, 4a\hat{x}, -2a\hat{x} \dots\} = \{(3m+1)a\hat{x} \mid m \in \mathbb{Z}\}$$

$$H + 2a\hat{x} = \{2a\hat{x}, -a\hat{x}, 5a\hat{x}, \dots\} = \{(3m+2)a\hat{x} \mid m \in \mathbb{Z}\}$$

$$T = H \cup (H + a\hat{x}) \cup (H + 2a\hat{x})$$

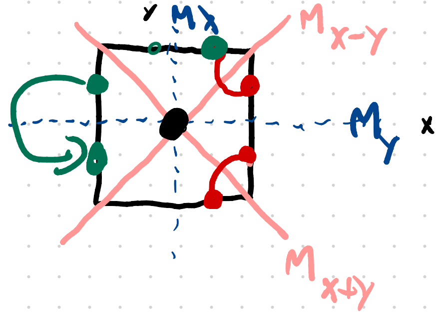
Conjugate H by $na\hat{x}$

$$na\hat{x} + H + (na\hat{x})^{-1} = \{\cancel{na\hat{x}} + 3ma\hat{x} - \cancel{na\hat{x}} \mid m \in \mathbb{Z}\}$$
$$= H$$

→ whenever a group is <u>abelian</u> (the binary operation is commutative) all subgroups are normal

Slightly less trivial example:

Symmetries of a square



90° rotation: C_{4z}

180° rotation $C_{4z}^2 = C_{2z}$

270° rotation $C_{4z}^3 = C_{4z}^{-1}$

E:

$\boxed{M_y}$: mirror reflection $y \to -y$

M_x: $x \to -x$

M_{x-y}: $x \to y$
$y \to x$

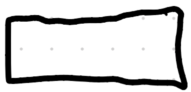
M_{x+y}: $x \to -y$
$y \to -x$

$\{E, M_x, M_y, M_{x-y}, M_{x+y},$
$C_{4z}, C_{2z}, C_{4z}^{-1}\}$ -
point group $4mm$

Conjugation: $C_{4z} M_x C_{4z}^{-1} = M_y$

Subgroups: $\{E, M_x\} < 4mm$

not a normal subgroup



Symmetry group of a rectangle

$\{E, M_x, M_y, C_{2z}\} \triangleleft 4mm$

normal subgroup

For normal subgroups, the product of two right cosets is also a right coset

$H \triangleleft G$ is normal

Hg_1, Hg_2 are right cosets

$$Hg_1Hg_2 = \{h \cdot g_1 \cdot h' \cdot g_2 \mid h, h' \in H\}$$

$$= \{h \cdot h'' g_1 g_2 \mid h, h'' \in H\}$$

$$= \{h g_1 g_2 \mid h \in H\}$$

$$= Hg_1g_2$$

$g_1 h' = h'' g_1$

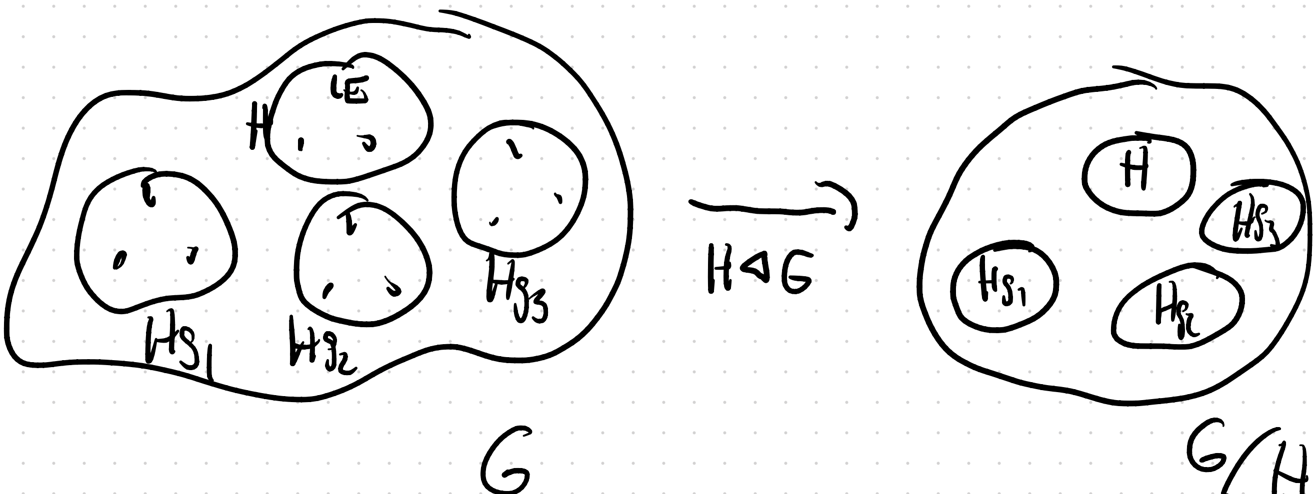
Since H normal

$HE = H$ $\quad$ $HHg_1 = Hg_1$ $\quad$ the subgroup acts like the identity

$$Hg_1 \cdot Hg_1^{-1} = H$$

$\Rightarrow$ For normal subgroups, the set of right cosets $\{H, Hg_1, Hg_2, \ldots, Hg_{n-1}\}$ forms a group!

The <u>quotient group</u> G/H



Ex: 1D Bravais lattice $\quad T = \{n a\hat{x} \mid n \in \mathbb{Z}\}$

$$H = \{0, 3a\hat{x}, -3a\hat{x}, \dots\} = \{3na\hat{x} \mid n \in \mathbb{Z}\}$$

$$H + a\hat{x} = \{a\hat{x}, 4a\hat{x}, -2a\hat{x}, \dots\} = \{(3m+1)a\hat{x} \mid m \in \mathbb{Z}\}$$

$$H + 2a\hat{x} = \{2a\hat{x}, -a\hat{x}, 5a\hat{x}, \dots\} = \{(3m+2)a\hat{x} \mid m \in \mathbb{Z}\}$$

$$H + H = H$$

$$(H + a\hat{x}) + H = H + a\hat{x}$$

$$(H + 2a\hat{x}) + H = H + 2a\hat{x}$$

$$(H + a\hat{x}) + (H + a\hat{x}) = H + 2a\hat{x}$$

$$(H + a\hat{x}) + (H + 2a\hat{x}) = H + 3a\hat{x} = H$$

$$(H + 2a\hat{x}) + (H + 2a\hat{x}) = H + 4a\hat{x} = H + a\hat{x}$$

$$H \sim [0]$$

$$H + a\bar{x} \sim [1]$$
$$H + 2a\bar{x} \sim [2]$$

$T/H \cong \mathbb{Z}_3$ integers w/ addition mod 3

In quantum mechanics

$$\underset{\text{group of symmetries}}{\phi : G} \longrightarrow \underset{\text{unitary operators on Hilbert space}}{K}$$

$$\phi : g \mapsto \phi(g) \in K$$

Subset of functions ϕ compatible w/ group multiplication

$$\phi(g_1 g_2) = \phi(g_1)\phi(g_2)$$ group homomorphism

$G \ni E_G$ - identity in G

$K \ni E_K$ - identity in K

Example: $T = \{ \sum_i n_i \vec{t}_i \mid n_i \in \mathbb{Z} \}$

K group of unitary operators on wavefunctions $\psi(\vec{x})$

$$T \downarrow$$

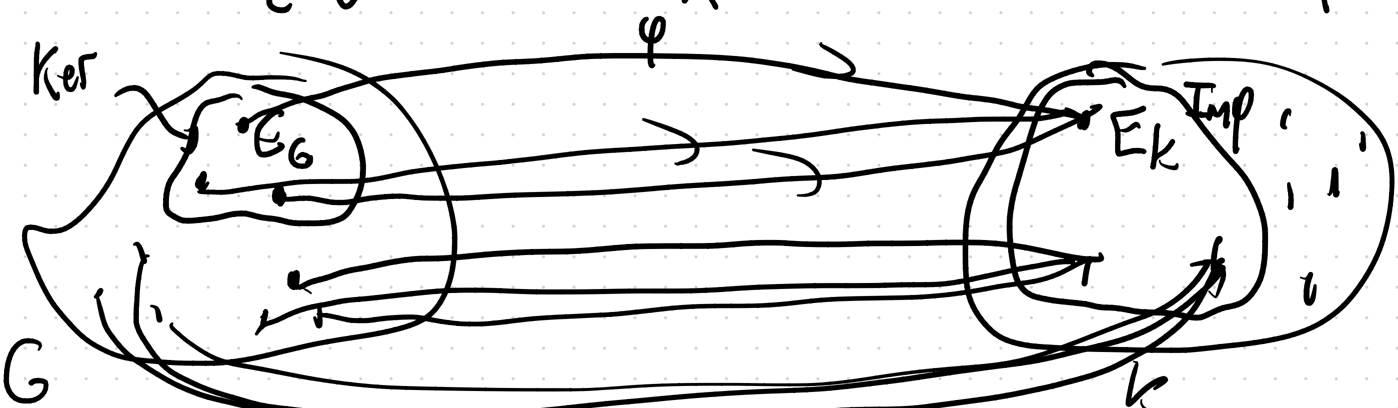
$$\phi\left(\sum_i n_i \vec{t}_i\right) = e^{-\frac{i}{\hbar}\vec{p}\cdot\left(\sum_i n_i t_i\right)}$$

$\vec{p} = -i\hbar \frac{\partial}{\partial \vec{x}}$ momentum operator

Given $\varphi: G \to K$

$Im(\varphi) = \{\varphi(g) \mid g \in G\} \subset K$ image of φ

$Ker(\varphi) = \{g \mid \varphi(g) = E_K\} \subset G$ kernel of φ



① $\mathrm{Im}\,\varphi \leq K$ is a subgroup

$E_K = \varphi(E_G) \in \mathrm{Im}(\varphi)$

$\varphi(g_1)\varphi(g_1^{-1}) = \varphi(E_G)$
$= E_K$

pf: $k_1 \in \mathrm{Im}(\varphi) \quad k_2 \in \mathrm{Im}\,\varphi$

$\downarrow \qquad \downarrow$

$k_1 = \varphi(g_1) \qquad k_2 = \varphi(g_2)$

$k_1 k_2 = \varphi(g_1)\varphi(g_2) = \varphi(g_1 g_2) \in \mathrm{Im}(\varphi)$

$k_1 = \varphi(g_1) \Rightarrow k_1^{-1} = [\varphi(g_1)]^{-1} = \varphi(g_1^{-1}) \in \mathrm{Im}(\varphi)$

(2) $\ker\varphi \trianglelefteq G$ is a normal subgroup of G

pf; (A) $\ker\varphi \leq G$ $\qquad g_1, g_2 \in \ker\varphi$

$$\varphi(g_1 g_2) = \varphi(g_1)\varphi(g_2) = E_K E_K = E_K$$

$$g_1 g_2 \in \ker\varphi$$

$$E_G \in \ker\varphi$$

$$\varphi(g_1) = E_K \qquad \varphi(g_1^{-1}) = [\varphi(g_1)]^{-1} = E_K$$

(B) $\quad g \in \ker\varphi \qquad g' \in G$

$$g' g g'^{-1} \overset{?}{\in} \ker\varphi$$

$$\varphi(g' g g'^{-1}) = \varphi(g')\underbrace{\varphi(g)}_{E_K}\varphi(g'^{-1}) = E_K$$

<u>First isomorphism theorem</u>: let G, K be groups, $\varphi: G \to K$ a homomorphism

$G/\ker\varphi$ is isomorphic to $\mathrm{Im}(\varphi)$

there exists a 1-to-1 (invertible) homomorphism between them