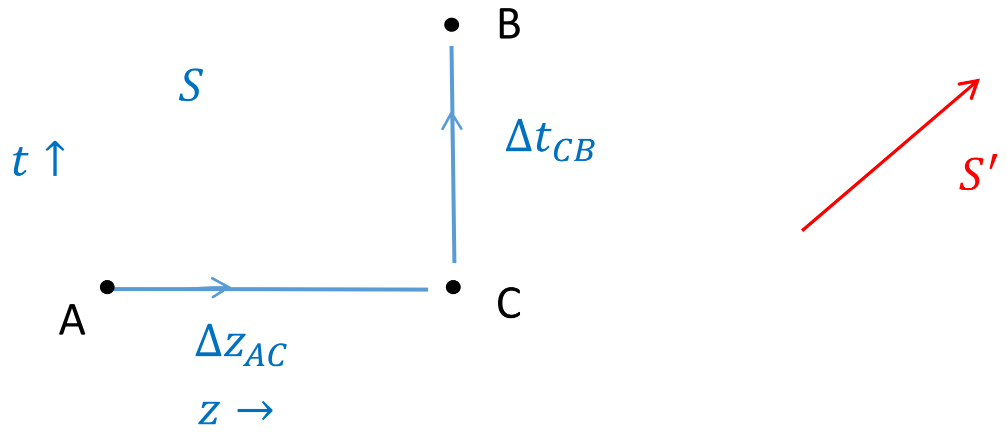
DERIVATION OF LORENTZ TRANSFORMATION



Problem: given Δz_{AB} and Δt_{AB}, what are $\Delta z'_{AB}$, $\Delta t'_{AB}$?

(1) For $\Delta z = 0, \Delta t \neq 0$ $\quad (1 - v^2/c^2)^{-1/2}$

$$\Delta t' = \gamma(v)\Delta t \leftarrow \text{Fitzgerald time dilation}$$

(2) Since S' sees S move in time $\Delta t'$ a distance $-v\Delta t'$

$$\Delta z'|_{\Delta z=0} = -v\Delta t'|_{\Delta z=0} = -v\gamma(v)\Delta t$$

(3) For $\Delta t = 0$, $\Delta z \neq 0$,

$$\Delta z'|_{\Delta t=0} = \gamma(v)\Delta z \qquad \leftarrow \text{inverted Lorentz contraction}$$

(4) For $\Delta t = 0$, $\Delta z \neq 0$,

$$\Delta t'|_{\Delta t=0} = \frac{-v\,\gamma(v)}{c^2}\Delta z \qquad \leftarrow \text{"relativity of simultaneity"}$$

In our case this implies:

by (3) and (4),

$$\Delta z'_{AC} = \gamma(v)\Delta z_{AC}, \qquad \Delta t'_{AC} = -(\gamma(v)\, v/c^2)\Delta z_{AC}$$

and by (1) and (2),

$$\Delta z'_{CB} = -v\gamma(v)\Delta t_{CB}, \Delta t'_{CB} = \gamma(v)\Delta t_{CB}$$

so since $\Delta z'_{AB} = \Delta z'_{AC} + \Delta z'_{CB}$ (etc.), and $\Delta z_{AC} \equiv \Delta z_{AB}$

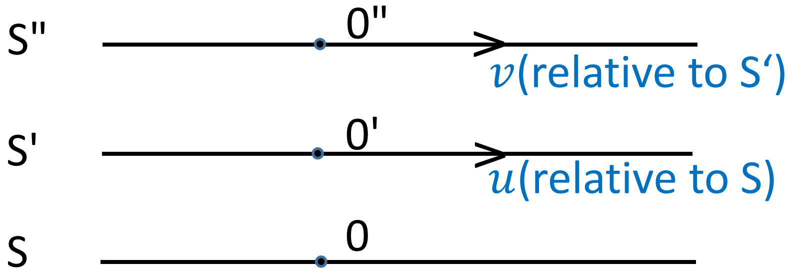
$$\Delta z'_{AB} = \gamma(v)(\Delta z_{AB} - v\Delta t_{AB})$$

$$\Delta t'_{AB} = \gamma(v)(\Delta t_{AB} - (v/c^2)\Delta z_{AB})$$

(and $\Delta x'_{AB} = \Delta x_{AB}$, etc.)

↑
standard Lorentz transformation

ADDITION OF VELOCITIES IN SPECIAL RELATIVITY



What is velocity of S" relative to S? (call it w)

Consider (any) two events as viewed from S, S' and S":

L.T. : (S → S')

$$\Delta x' = \frac{\Delta x - u\Delta t}{\sqrt{1 - u^2/c^2}} \qquad \Delta t' = \frac{\Delta t - u\Delta x/c^2}{\sqrt{1 - u^2/c^2}}$$

L.T. : (S' → S")

$$\Delta x'' = \frac{\Delta x' - v\Delta t'}{\sqrt{1 - v^2/c^2}} \qquad \Delta t'' = \text{[not needed]}$$

$$\equiv \frac{(\Delta x - u\Delta t) - v(\Delta t - u\Delta x/c^2)}{\sqrt{(1 - u^2/c^2)(1 - v^2/c^2)}}$$

$$\equiv \frac{\Delta x(1 + uv/c^2) - (u + v)\Delta t}{\sqrt{\cdots}}$$

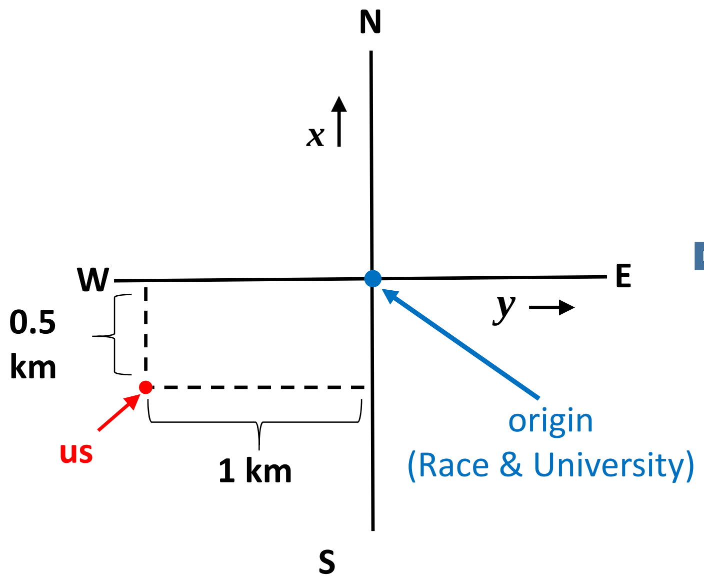
Suppose 2 events are e.g. light flashes emitted by 0", so $\Delta x'' = 0$, then $\Delta x/\Delta t = (u + v)/(1 + uv/c^2)$. But $\Delta x/\Delta t$ is by definition just w, so

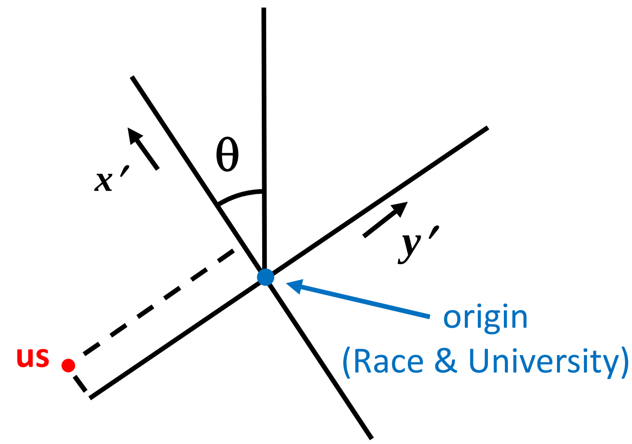
$$\boxed{w = \frac{u + v}{1 + uv/c^2}}$$

Which is $< c$ for any $u, v < c$

MINKOWSKI DIAGRAM

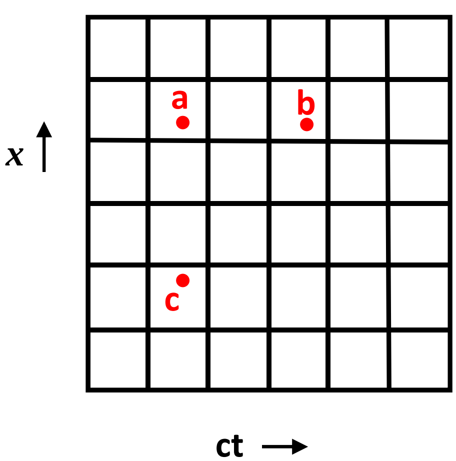
Analogy: rotation in ordinary (2D) space

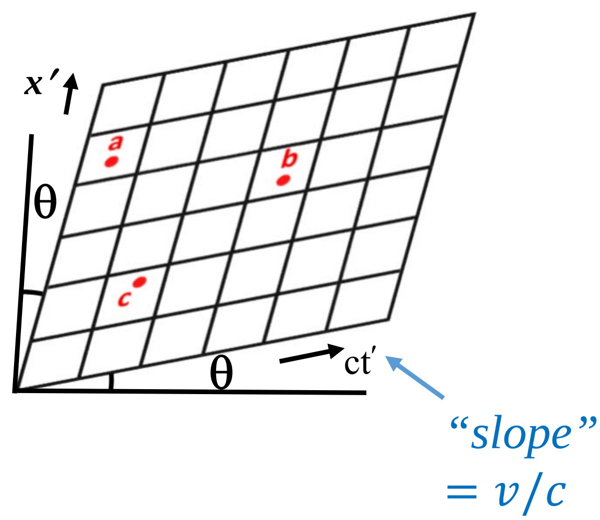




$x' \neq x, y' \neq y$ (but $(\text{distance})^2 = x'^2 + y'^2 = x^2 + y^2$)

"Minkowski space":





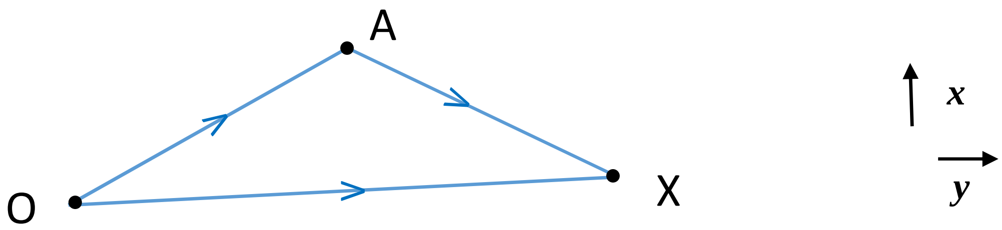
$x' \neq x, t' \neq t$

$$\underbrace{c^2t'^2 - x'^2}_{(\Delta s')^2} = \underbrace{c^2t^2 - x^2}_{(\Delta s)^2}$$

provided we rescale axes appropriately

THE TWINS PARADOX IN MINKOWSKI SPACE*:

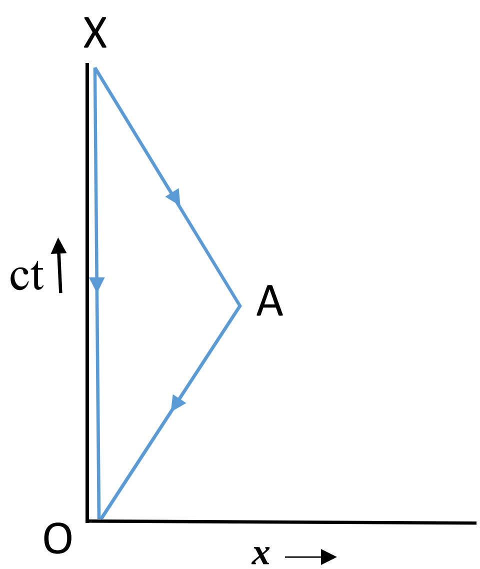
Analogy in ordinary (2D) space: (trivial!)



$(\Delta s)^2_{2D} = (\Delta x)^2 + (\Delta y)^2$

$\Delta s_{O \rightarrow A \rightarrow X} = \Delta s_{OA} + \Delta s_{AX} \neq \Delta s_{OX}$ (>)

Minkowski space:



(viewed from Alice's (inertial) frame)

$(\Delta s)^2_{Min} = c^2(\Delta t)^2 - (\Delta x)^2$

Proper time elapsed = $(\Delta s_{total})/c$

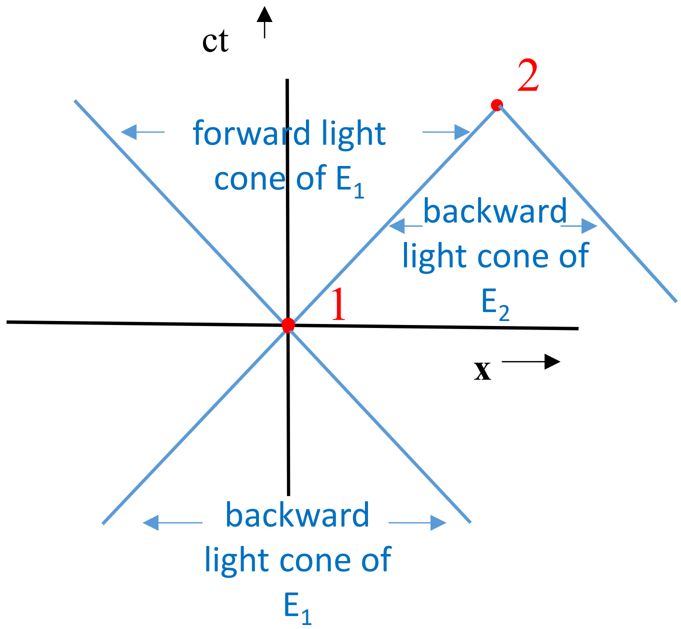
$(\Delta s_{total})_{Barbon} = \Delta s_{OA} + \Delta s_{AX} \neq \Delta s_{OX} = ct$ (<)

*largely based on A.P. French, Special Relativity. pp. 154 - 9

(a) $\Delta s_{12}^2 = 0$

("light-like separated")

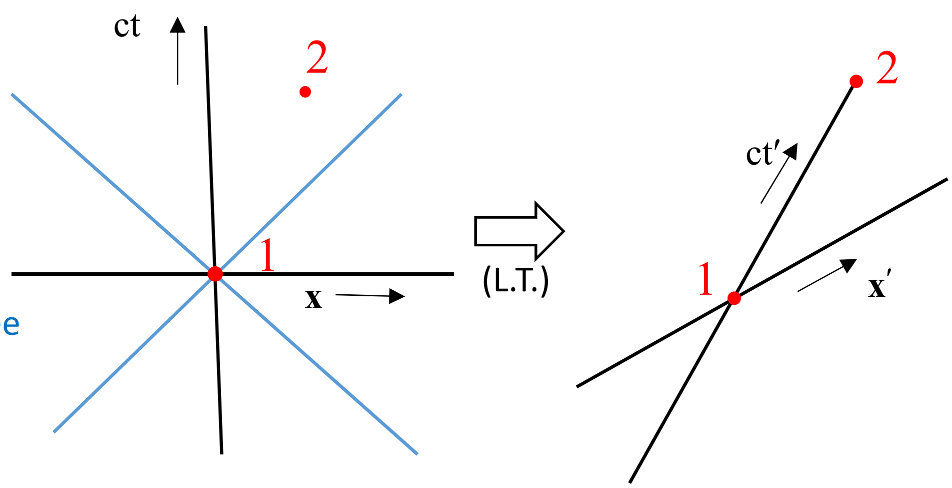
All observers agree on sign of Δt_{12}



(b) $\Delta s_{12}^2 > 0$

("timelike separated")

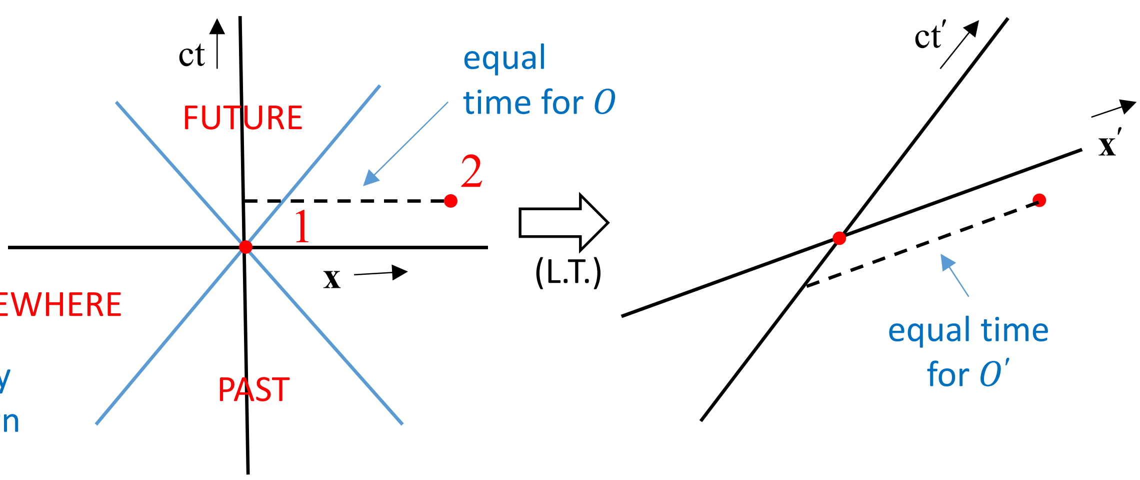
All observers still agree on sign of Δt_{12}

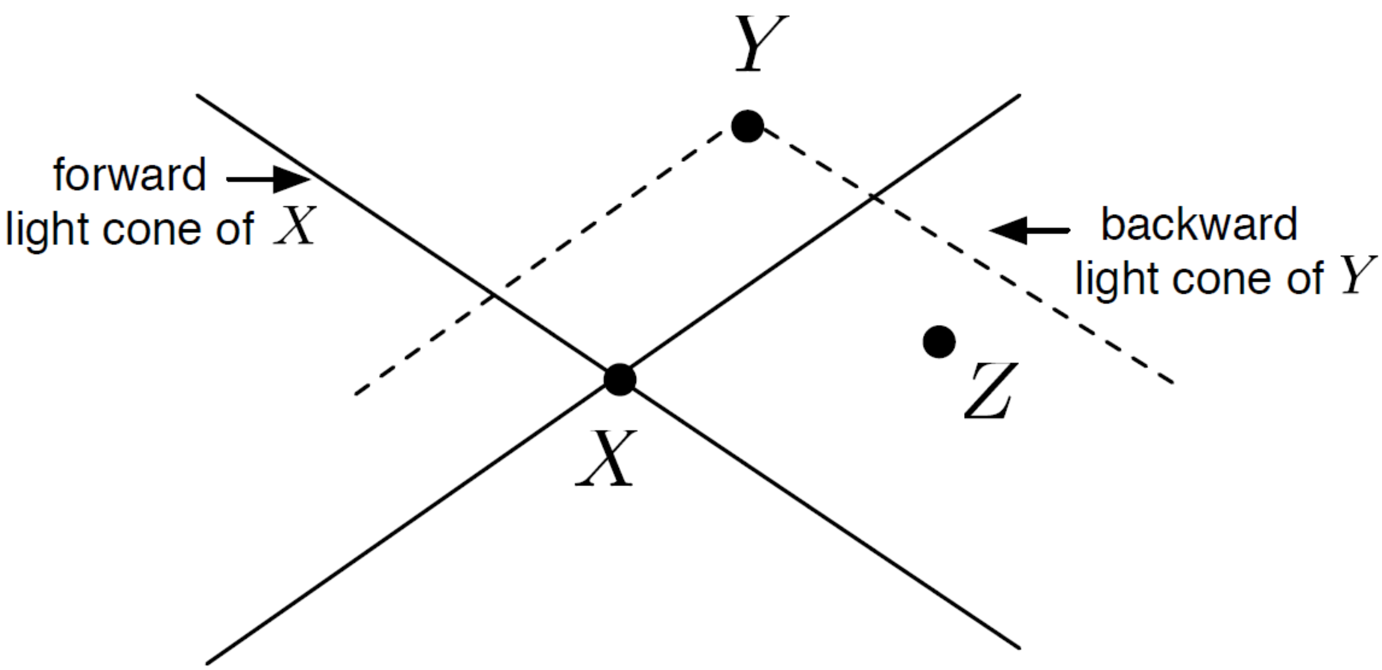


(c) $\Delta s_{12}^2 < 0$

("spacelike separated")

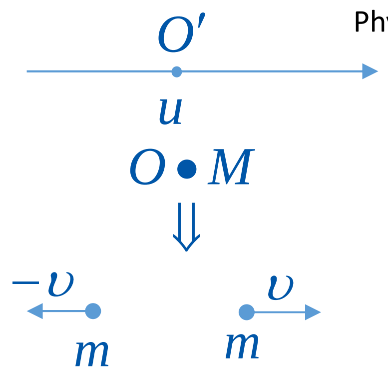
Observers may disagree on sign of Δt_{12}




Y
forward
light cone of X
backward
light cone of Y
Z
X

Relativistic mechanics

	Newton	Einstein
Transformation	$t' = t$, $x' = x - \upsilon t$	$t' = \gamma(\upsilon)\left(t - \left(\upsilon / c^2\right)x\right)$, $x' = \gamma(\upsilon)(x - \upsilon t)$
Invariant?		
time interval	yes	no
spatial interval	no	no
space-time interval	no	yes
mass	yes	yes
momentum	no	no
energy	no	no
Addition of velocities	$\upsilon_{A-C} = \upsilon_{A-B} + \upsilon_{B-C}$	$\upsilon_{A-C} = \dfrac{\upsilon_{A-B} + \upsilon_{B-C}}{1 + \left(\upsilon_{A-B}\upsilon_{B-C} / c^2\right)}$



O reckons:

Initial momentum $\rightarrow P = 0$

Final momentum $\rightarrow P' = m\upsilon + m(-\upsilon) = 0$

$\Rightarrow P' = P,$ momentum is conserved

O' reckons:

$$P = -Mu = -2mu$$

$$P' = -m\upsilon_1' + m\upsilon_2'$$

but if O' uses relativistic velocity-addition law, then he reckons

$$\upsilon_1' = \frac{-u + \upsilon_1}{1 + u\upsilon_1 / c^2} \equiv \frac{-u + \upsilon}{1 + u\upsilon / c^2} \qquad \upsilon_2' = \frac{-u + \upsilon_2}{1 - u\upsilon_2 / c^2} = \frac{-u - \upsilon}{1 - u\upsilon / c^2}$$

and so

$$P' = m\left(\frac{-u + \upsilon}{1 + u\upsilon / c^2}\right) + m\left(\frac{-u - \upsilon}{1 - u\upsilon / c^2}\right) = -2mu\left(\frac{1 - \upsilon^2 / c^2}{1 - (u\upsilon)^2 / c^4}\right) \neq P!$$

momentum is not conserved $\Rightarrow$ Lorentz invariance violated.

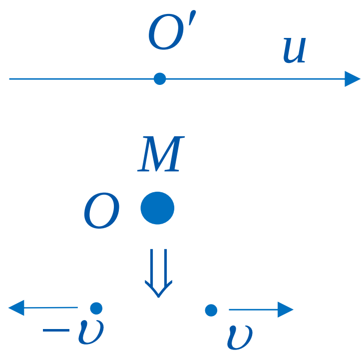
(since statement true for O is not true for O')

Conclusion: cannot simultaneously maintain

(1) Conservation of mass

(2) Newton's laws with standard definition of momentum

(3) General Lorentz invarance



Relax mass conservation, i.e. set $M \neq 2m$?
But would need

$$M/2m = \frac{\left(1-\upsilon^2/c^2\right)}{1-\left(u\upsilon/c^2\right)^2} \qquad \leftarrow \text{depends on u}$$

$\Rightarrow$ violates Lorentz invariance

Alternatively, consider collision where all masses same, then

O reckons:

$$P_{in} = \sum_i m_i \upsilon_{in}^{(i)} = \sum_i m_i \upsilon_{out}^{(i)} \equiv P_{out}$$

O' reckons:

$$P'_{in} = \sum_i m_i \upsilon_{in}'^{(i)} = \sum_i m_i \frac{\left(\upsilon_{in}^{(i)} - u\right)}{1 + u\upsilon_{in}^{(i)}/c^2}$$

$$P'_{out} = \sum_i m_i \upsilon_{out}'^{(i)} = \sum_i m_i \frac{\left(\upsilon_{out}^{(i)} - u\right)}{\left(1 + u\upsilon_{out}^{(i)}/c^2\right)}$$

in general,

$$\left.\begin{aligned} P'_{out} &\neq P'_{in} \\ P_{out} &= P_{in} \end{aligned}\right\} \Rightarrow \text{Violate Lorentz invarance}$$

thus,

$$P' \equiv \sum_i m_i J_i = \frac{1}{\sqrt{1-u^2/c^2}} \sum_i \frac{m_i\left(\upsilon_i - u\right)}{\sqrt{1-\upsilon_i^2/c^2}}$$

$$= \frac{1}{\sqrt{1-u^2/c^2}} \left\{ \sum_i \frac{m_i \upsilon_i}{\sqrt{1-\upsilon_i^2/c^2}} - u \sum_i \frac{m_i c^2}{\sqrt{1-\upsilon_i^2/c^2}} \right\}$$

↑	↑	↑
constant	conserved by hypothesis	must also be conserved!

If we assume that description of collision processes is Lorentz invariant, i.e. that conservation of momentum in one inertial frame implies its conservation in any other, then we are forced to conclude that the quantity

$$\sum_i \frac{m_i c^2}{\sqrt{1-\upsilon_i^2/c^2}} \qquad (*)$$

is conserved in any inertial frame.

Now, $$\frac{mc^2}{\sqrt{1-\upsilon^2/c^2}} = mc^2 + \frac{1}{2}m\upsilon^2 + \ldots$$

↑ nonrelativisitic KE

Should we take the mc^2 seriously?

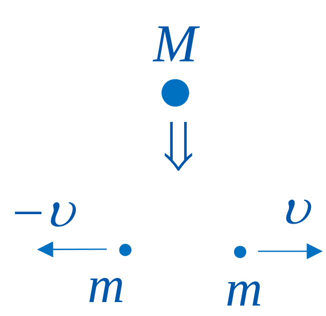
One advantage if we do: On Lorentz transformation, then it turns out that

$$P' = \gamma(\upsilon)\left(P - \upsilon E/c^2\right)$$

$$E' = \gamma(\upsilon)\left(E - \upsilon P\right)$$

Just like x and t!

Disintegration:



Conservation of the expansion (*) implies

$$Mc^2 = 2mc^2/\sqrt{1-\upsilon^2/c^2}$$

or

$$M = \frac{2m}{\sqrt{1-\upsilon^2/c^2}} \neq 2m$$

Momentum seen by observer O':

$$P' = \sum_i \frac{m_i \upsilon_i'}{\sqrt{1 - \upsilon_i'^2 / c^2}}$$

but $\upsilon_i' = \dfrac{\upsilon_i - u}{1 - u\upsilon_i / c^2}$, so

$$P' = \sum_i m_i J_i \text{ , when}$$

$$J_i \equiv \frac{(\upsilon_i - u)/\left(1 - u\upsilon_i / c^2\right)}{\sqrt{1 - \dfrac{(\upsilon_i - u)^2 / c^2}{\left(1 - u\upsilon_i / c^2\right)^2}}} \quad \times \div \; 1 - u\upsilon_i / c^2 :$$

$$J_i = \frac{\upsilon_i - u}{\sqrt{\left(1 - u\upsilon_i / c^2\right)^2 - (\upsilon_i - u)^2 / c^2}}$$

$$= \frac{\upsilon_i - u}{\sqrt{1 - c^2 + u^2\upsilon_i^2 / c^4 - \upsilon_i^2 / c^2 + /c^2 - u^2 / c^2}}$$

$$= \frac{\upsilon_i - u}{\sqrt{1 + u_i^2\upsilon_i^2 / c^4 - \upsilon_i^2 / c^2 - u^2 / c^2}} = \frac{\upsilon_i - u}{\sqrt{1 - \upsilon_i^2 / c^2} \cdot \sqrt{1 - u^2 / c^2}}$$