PHSY 515: Homework Set 11

April 21, 2020

Due Date: Tuesday April 28, 2020, at the beginning of class.

Topic: Graduate General Relativity 1

1. In this problem you will go through the entire derivation of the Schwarzschild metric in Kruskal coordinates and learn to understand the Kruskal diagrams and conformal diagrams. This is all presented in chapter 5 of Carroll and thus it will not be very difficult to do. However, you will be expected to reproduce the results on your own and explain every step and meaning in your own words with sufficient detail.

   a) Starting with Eq. (5.115) in Carroll Derive Eq. (5.116) and explain why we do this transformation.

   b) Then make the same arguments as Carroll to come to (5.120) and explain in your own words why we would want to do this.

   c) Now change to the Kruskal coordinates and use (5.121) and (5.120) to arrive at (5.123).

   d) Explain the properties that follow (5.123) appearing in the book in your own words and construct a neatly drawn and very well labeled Kruskal diagram. Your diagram should include:

      i) the singularities,

      ii) the Horizons,

      iii) lines of constant r and t (be reasonable with how many you draw, meaning not just one or two and not 10),

      iv) and the light cones.

   e) Explain the four regions of this diagram and what they mean. Do not be short. Explain it clearly and well enough so that I know it is obvious you know what your are talking about.

   f) Address the "wormhole" issue described in class and in the book and explain why you can't travel through the "hole" in spacetime. Use drawings and very clear explanation to make your case.

   g) Read Appendix H in Carroll and clearly explain the usefulness of conformal diagrams in GR.

   h) Construct a conformal diagram for Schwarzschild (like the one in figure 5.16) that is neatly drawn and clearly labeled. From this diagram explain what every symbol in the diagram means. Appendix H should help in this regard.

   i) Understand the last paragraph in Chapter 5.7 and explain in sufficient detail the meaning of i^+ and i^-.

2. In this problem you will derive the TOV equations and begin setting up the problem so that you can numerically integrate the equations, but you will not be asked to integrate them.

   a) Starting from the general static, spherically symmetric metric and assuming a perfect fluid stress-energy tensor, derive the TOV equation like done in class. In the end you should have a differential equation for dm/dr and dp/dr.

   b) One case (potentially the only case) in which the TOV equation can be solved analytically is when you have an incompressible fluid, i.e. $\rho =$ constant. Solve the TOV equations analytically for this case and explain the significance of Buchdahl's theorem (presented in Carroll chapter 5.8) in sufficient detail.

   c) If you want to solve the TOV equations for any other equation of state then you will have to do so numerically. One problem with this, as mentioned in class, is that the equations are divergent at $r = 0$. This means that when integrating these equations we cannot start our integrations here. We also cannot just start the integration at an arbitrarily small value of radius and assume that the same boundary conditions hold. The way around this issue is to do a power series of your variables around $r = 0$, i.e.

$$\begin{aligned} m(r) &= \sum_j m_j r^j \ , \\ p(r) &= \sum_j p_j r^j \ , \\ \rho(r) &= \sum_j \rho_j r^j \ . \end{aligned}$$

Assume that the equation of state has the expansion near the central density that is

$$p = p(\rho_c) + (p_c \Gamma_c / \rho_c)(\rho - \rho_c) + \cdots$$

with the constant Γ_c being the adiabatic index $d(\ln p)/d(\ln \rho)$ evaluated at ρ_c. For the purposes of this problem, find the first two non-vanishing terms in each power series above and estimate the largest radius r at which these terms give an error no larger than 0.1% in any of the power series. These power series allow us to accurately start our numerical integrations away from $r = 0$ by accurately calculating new boundary conditions for $r = r_c$ where we start the integration.

3. Show that the trajectory of light rays in Schwarzschild obeys

$$\frac{d^2 u}{d\phi^2} + u = 3u^2 \ ,$$

where $u = M/r$ and r is the usual Schwarzschild radius. Denote the minimum value of r along the trajectory by b, which is known as the impart parameters. In the case that $M/b \ll 1$ what is the deflection of a photon as it passes a spherical gravitating body? Give a formula for the deflection angle to lowest non-vanishing order in M/b.

4. Prove that the metric,

$$ds^2 = -dt^2 + \frac{4}{9}\left[\frac{9M}{2(r-t)}\right]^{2/3} dr^2 + \left[\frac{9M}{2}(r-t)^2\right]^{2/3} d\Omega^2$$

is actually static. Show that it is in fact the Schwarzschild geometry. Now show that the set of coordinate-stationary observers are all in free fall and have zero energy. This is kind of similar to a problem you had in the previous homework.

5. A certain scalar field satisfies $\Box\Phi = 0$. Show that in the Schwarzschild geometry Φ can be decomposed in spherical harmonic components as

$$\Phi = \frac{1}{r}\psi(r,t) Y_{\ell m}(\theta,\phi) \ ,$$

where ψ satisfies

$$\psi_{,tt} = \left(1 - \frac{2M}{r}\right)\left[\left(1 - \frac{2M}{r}\right)\psi_{,r}\right]_{,r} - V_\ell(r)\psi \ ,$$

with

$$V_\ell(r) = \left(1 - \frac{2M}{r}\right)\left[\frac{2M}{r^3} + \frac{\ell(\ell+1)}{r^2}\right] \ .$$

This problem is not that bad. Just write out $\Box\Phi$ using the Schwarzschild metric and plug in the desired solution to show that you can actually decompose the field this way.

6. One of the popular extensions of GR is Brans-Dicke gravity where there exists a scalar field coupled to matter. In a general we can write the field equations in Brans-Dicke gravity as

$$\begin{aligned} G^{\mu\nu} + F^{\mu\nu}(\Phi_{;\alpha}\,,\,\Phi_{;\alpha\beta}) &= T^{\mu\nu} \ , \\ \Box\Phi &= T^{\mu}{}_{\mu} \ , \end{aligned}$$

where Φ is the scalar field and $F^{\mu\nu}$ is a tensor that only depends on first and second derivatives of the scalar field. All other quantities are the same as they appear in GR. Calculate the general static, spherically symmetric vacuum solution to these field equations assuming that the scalar field does not diverge anywhere.