

PHYS 515: Homework Set 10

April 17, 2020

Due Date: Tuesday April 21, 2020, at the beginning of class.

Topic: Graduate General Relativity 1

1. Carroll 5.3

Consider a particle (not necessarily on a geodesic) that has fallen inside the event horizon, $r < 2GM$. Use the ordinary Schwarzschild coordinates (t, r, θ, ϕ) . Show that the radial coordinate must decrease at a minimum rate given by

$$\left| \frac{dr}{d\tau} \right| \geq \sqrt{\frac{2GM}{r} - 1}.$$

Calculate the maximum lifetime for a particle along a trajectory from $r = 2GM$ to $r = 0$. Express this in seconds for a black hole with mass measured in solar masses. Show that this maximum proper time is achieved by falling freely with $E \rightarrow 0$.

2. Carroll 5.4b

Consider Einstein's equations in vacuum, but with a cosmological constant, $G_{\mu\nu} + \Lambda g_{\mu\nu} = 0$. Write down the equation of motion for radial geodesics in terms of an effective potential, as in (5.66). Sketch the effective potential for massive particles.

3. Carroll 5.5

Consider a comoving observer sitting at constant spatial coordinates (r_*, θ_*, ϕ_*) , around a Schwarzschild black hole of mass M . The observer drips a beacon into the black hole (straight down, along a radial trajectory). The beacon emits radiation at a constant wavelength λ_{em} (in the beacons rest frame).

- Calculate the coordinate speed dr/dt of the beacon, as a function of r .
- Calculate the proper speed of the beacon. That is, imagine there is a comoving observer at fixed r , with a locally inertial coordinate system set up as the beacon passes by, and calculate the speed as measured by the comoving observer. What is it at $r = 2GM$.
- Calculate the wavelength λ_{obs} , measured by the observer at r_* , as a function of the radius r_{em} at which the radiation was emitted.
- Calculate the time t_{obs} at which a beam emitted by the beacon at radius r_{em} will be observed at r_* .
- Show that at late times, the redshift grows exponentially: $\lambda_{\text{obs}}/\lambda_{\text{em}} \propto e^{t_{\text{obs}}/T}$. Give an expression for the same time constant T in terms of the black hole mass M .

4. Poisson and Will 5.18

In the course of your study of general relativity you come across a vacuum solution to the Einstein field equations given by

$$ds^2 = -d(ct)^2 + \left(\frac{4\alpha ct x^j}{r(r^2 - \alpha^2)} \right) d(ct) dx^j + \left[\frac{(r + \alpha)^4}{r^4} \delta_{jk} - \frac{4\alpha^2 c^2 t^2 x^j x^k}{r^2(r^2 - \alpha^2)^2} \right] dx^j dx^k,$$

in which α is a constant and $r^2 := \delta_{jk} x^j x^k$. You take it upon yourself to study the significance of this spacetime.

- a) Transform the metric from Cartesian coordinates x^j to the standard spherical symmetrical polar coordinates (r, θ, ϕ) , and show that the metric is, in fact, spherically symmetric. *Hint:* What are $\delta_{jk}x^jx^k$ and $(x^j/r)dx^j$ in spherical coordinates?
- b) Calculate the acceleration of a body at rest at very large r , and use your result to relate the parameter α to the total mass M in the spacetime.
- c) Find a coordinate transformation that puts the metric in static form, and confirm your result in part b) by reading off the mass directly from the metric.
- d) Can you name this spacetime?

5. Schutz 11.7

A clock is in a circular orbit at $r = 10M$ in a Schwarzschild metric.

- a) How much time elapses on the clock during one orbit? (Integrate proper time $d\tau = \sqrt{-ds^2}$ over an orbit.)
- b) It send out a signal to a distant observer once each orbit. What time interval does the distant observer measure between receiving any two signals?
- c) A second clock is located at rest at $r = 10M$ next to the orbit of the first clock. (Rockets keep it there.) How much time elapses on it between successive passes of the orbiting clock?
- d) Calculate part b) again in second for an orbit at $r = 6M$ where $M = 14M_\odot$. This is the minimum fluctuation time we expect in the X-ray spectrum of Cyg X-1: Why?
- e) If the orbiting 'clock' is twin Artemis, in orbit in part d), how much does she age during the time her twin Diana lives 40 years far from the black hole and at rest with respect to it?