PHYS 515: Homework Set 9

April 7, 2020

Due Date: Tuesday April 14, 2020, at the beginning of class.

Topic: Graduate General Relativity 1

1. Consider Einstein's equations in vacuum, but with a cosmological constant, $G_{\mu\nu} + \Lambda g_{\mu\nu} = 0$. Solve for the most general spherically symmetric metric, in coordinates (t, r) that reduce to the ordinary Schwarzschild coordinates when $\Lambda = 0$.

2. Let M be a three-dimensional manifold possessing a spherically symmetric Riemannian metric with $\nabla_\alpha r \neq 0$, where r is defined by $r = (A/4\pi)^{1/2}$ with A being the total area of a two-sphere.

   a) Show that a new "isotropic" radial coordinate $\tilde{r}$ can be introduced so that the metric takes the form $ds^2 = H(\tilde{r})[d\tilde{r}^2 + \tilde{r}^2 d\Omega^2]$. (This shows that every spherically symmetric three-dimensional space is conformally flat.)

   b) Show that in isotropic coordinates the Schwarzschild metric is

$$ds^2 = -\frac{(1 - M/2\tilde{r})^2}{(1 + M/2\tilde{r})^2} dt^2 + \left(a + \frac{M}{2\tilde{r}}\right)^4 [d\tilde{r}^2 + \tilde{r}^2 d\Omega^2] \ .$$

3. Consider the course-free ($j^\mu = 0$) Maxwell's equations

$$\nabla^\mu F_{\mu\nu} = 0 \ ,$$

$$\nabla_{[\rho} F_{\mu\nu]} = 0 \ ,$$

   in a static, spherically symmetric spacetime described by the line element

$$ds^2 = -f(r)dt^2 + h(r)dr^2 + r^2 d\Omega^2 \ .$$

   a) Argue that the general form of a Maxwell tensor which shares the static and spherical symmetries of the spacetime is $F_{\mu\nu} = 2A(r)(\mathbf{e}_{(0)})_{[\mu}(\mathbf{e}_{(1)})_{\nu]} + 2B(r)(\mathbf{e}_{(2)})_{[\mu}(\mathbf{e}_{(3)})_{\nu]}$, where $A(r)$ and $B(r)$ are just functions and $\mathbf{e}_{(\mu)}$ are the basis vectors associated with the metric.

   b) Show that if $B(r) = 0$, the general solution of Maxwell's equations with the form of part a) is $A(r) = -q/r^2$, where q may be interpreted as the total charge.

   c) Write down and solve Einstein's equations, $G_{\mu\nu} = 8\pi T\mu\nu$, with electromagnetic stress-energy tensor corresponding to the solution of part b). Show that the general solution is the *Reissner-Nordstrom metric*,

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{q^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega^2 \ .$$