PHYS 515: Homework Set 6

Due Date: Thursday March 26, 2020, at the beginning of class.

Topic: Graduate General Relativity 1

1. Carroll 3.5

   Consider a 2-sphere with coordinates (θ, ϕ) and metric

   $$ds^2 = d\theta^2 + \sin^2\theta d\phi^2 \ .$$

   a) Show that lines of constant longitude (ϕ= constant) are geodesics, and that the only line of constant latitude (θ = constant) that is a geodesic is the equator ($\theta = \pi/2$).

   b) Take a vector with components $V^\mu = (1, 0)$ and parallel-transport it once around a circle of constant latitude. What are the components of the resulting vectors, as a function of θ?

2. Carroll 3.6

   A good approximation to the metric outside the surface of Earth is provided by

   $$ds^2 = -(1 + 2\Phi)dt^2 + (1 - 2\Phi)dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2) \ ,$$

   where

   $$\Phi = -\frac{GM}{r}$$

   may be thought of as the familiar Newtonian gravitational potential. Here G is Newton's constant and M is the mass of the Earth. For this problem Φ may be assumed to be small.

   a) Imagine a clock on the surface of the Earth at distance R_1 from the Earth's center, and another clock on a tall building at distance R_2 from the Earth's center. Calculate the time elapsed on each clock as a function of the coordinate time t. Which clock moves faster?

   b) Solve for a geodesic corresponding to a circular orbit around the equator of the Earth ($\theta = \pi/2$). What is $d\phi/dt$?

   c) How much proper time elapses while a satellite at radius R_1 (skimming along the surface of the Earth, neglecting air resistance) completes one orbit? You can work to first order in Φ if you like. Plug in the actual number for the radius of the Earth and so on (don't forget to restore the speed of light) to get an answer in seconds. How does this number compare to the proper time elapsed on the clock stationary on the surface?

3. Carroll 3.8

   The metric for the three-sphere in coordinates $x^\mu = (\psi, \theta, \phi)$ can be written

   $$ds^2 = d\psi^2 + \sin^2\psi(d\theta^2 + \sin^2\theta d\phi^2) \ .$$

   a) Calculate te Christoffel connection coefficients. Use whatever method you like (but still by hand), but it is good practice to get the connection coefficients by varying the integral in (3.49).

   b) Calculate the Riemann tensor, Ricci tensor, and Ricci scalar.

   c) Show that (3.191) is obeyed by this metric, confirming that the three-sphere is a maximally symmetric space (as you would expect).

4. Carroll 3.9

   Show that the Weyl tensor $C^\mu{}_{\nu\rho\sigma}$ is left invariant by a conformal transformation.

5. Carroll 3.10

   Show that, for $n \geq 4$, the Weyl tensor satisfies a version of the Bianchi identity,

   $$\nabla_\rho C^\rho{}_{\sigma\mu\nu} = 2\frac{(n-3)}{(n-2)}\left(\nabla_{[\mu}R_{\nu]\sigma} + \frac{1}{2(n-1)}g_{\sigma[\mu}\nabla_{\nu]}R\right) \ .$$