PHYS 515: Homework Set 5

Due Date: Tuesday March 10, 2020, at the beginning of class.

Topic: Graduate General Relativity 1

1. Carroll 3.1

   Verify these consequences of metric compatibility ($\nabla_\sigma g_{\mu\nu} = 0$):

$$\begin{aligned} \nabla_\sigma g^{\mu\nu} &= 0 \; , \\ \nabla_\lambda \epsilon_{\mu\nu\rho\sigma} &= 0 \; . \end{aligned}$$

2. Carroll 3.3

   Imagine we have a *diagonal* metric $g_{\mu\nu}$. Show that the Christoffel symbols are given by

$$\begin{aligned} \Gamma^\lambda_{\mu\nu} &= 0 \; , \\ \Gamma^\lambda_{\mu\mu} &= -\frac{1}{2}(g_{\lambda\lambda})^{-1}\partial_\lambda g_{\mu\mu} \; , \\ \Gamma^\lambda_{\mu\lambda} &= \partial_\mu \left(\ln \sqrt{|g_{\lambda\lambda}|}\right) \; , \\ \Gamma^\lambda_{\lambda\lambda} &= \partial_\lambda \left(\ln \sqrt{|g_{\lambda\lambda}|}\right) \; . \end{aligned}$$

   In these expressions, $\mu \neq \nu \neq \lambda$, and repeated indices are *not* summed over.

3. Carroll 3.2

   You are familiar with the operations of gradient ($\nabla\phi$), divergence ($\mathbf{\nabla} \cdot \mathbf{V}$), and curl ($\mathbf{\nabla} \times \mathbf{V}$) in ordinary vector analysis in three-dimensional Euclidean space. Using covariant derivatives, derive formulae for these operations in spherical polar coordinates r, θ, ϕ defined by

$$\begin{aligned} x &= r \cos\theta \sin\phi \; , \\ y &= r \sin\theta \sin\phi \; , \\ z &= r \cos\theta \; . \end{aligned}$$

   Compare your results to those in Jackson (1999) or an equivalent text. Are they identical? Should they be?

4. Carroll 3.4

   In Euclidean three-space, we can define paraboloidal coordinates (u, v, ϕ) via

$$x = u\,v \cos\phi \qquad y = u\,v \sin\phi \qquad z = \frac{1}{2}(u^2 - v^2) \; .$$

   a) Find the coordinate transformation matrix between paraboloidal and Cartesian coordinates $\partial x^\alpha / \partial x^{\beta'}$ and the inverse transformation. Are there any singular points in the map?
   b) Find the basis vectors and basis one-forms in terms of Cartesian basis vectors and forms.
   c) Find the metric and inverse metric in paraboloidal coordinates.
   d) Calculate the Christoffel symbols.
   e) Calculate the divergence $\nabla_\mu V^\mu$ and Laplacian $\nabla_\mu \nabla^\mu f$.

5. Schutz 5.11

   For the vector field $\vec{V}$ whose Cartesian components are $(x^2 + 3y,\, y^2 + 3x)$

   a) Compute $V^\alpha{}_{,\beta}$ in Cartesian.
   b) Compute the transformation $\dfrac{\partial x^{\mu'}}{\partial x^\alpha}$ from Cartesian to polar, and its inverse $\dfrac{\partial x^\alpha}{\partial x^{\mu'}}$.

c) Compute the transformation $\frac{\partial x^{\mu'}}{\partial x^{\alpha}} \frac{\partial x^{\beta}}{\partial x^{\nu'}} V^{\alpha}{}_{,\beta}$ to polars.

d) Find the components of $V^{\mu'}{}_{;\nu'}$ directly in polars using the Christoffel symbols (calculate them, then use them).

e) Compute the divergence $V^{\alpha}{}_{,\alpha}$ using your results from part a).

f) Compute the divergence $V^{\mu'}{}_{;\mu'}$ using your results in either c) or d).

g) Compute the divergence $V^{\mu'}{}_{;\mu'}$ using

$$V^{\mu'}{}_{;\mu'} = \frac{1}{r}\partial_r(rV^r) + \partial_\theta V^\theta \ .$$