PHYS 515: Homework Set 3

Due Date: Tuesday February 25, 2020, at the beginning of class.

Topic: Graduate General Relativity 1

1. Carroll 1.8

   If $\partial_\nu T^{\mu\nu} = Q^\mu$, what physically does the spatial vector Q^i represent? Use the dust energy momentum tensor to make your case.

2. Carroll 1.9

   For a system of discrete point particles the energy-momentum tensor takes the form

$$T_{\mu\nu} = \sum_a \frac{p_\mu^{(a)} p_\nu^{(a)}}{p^{0(a)}} \delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}^{(a)}) \ ,$$

   where the index a labels the different particles. Show that, for a dense collection of particles with isotropically distributed velocities, we can smooth over the individual particle worldlines to obtain the perfect-fluid energy-momentum tensor (1.114).

3. Schutz 4.24

   Astronomical observations of the brightness of objects are measurements of the flux of radiation T^{01} from the object at Earth. This problem calculates how that flux depends on the relative velocity of the object and Earth.

   a) Show that, in the rest frame $\mathcal{O}$ of a star of constant luminosity L (total energy radiated per second), the stress-energy tensor of the radiation from the star at the event $(t,\, x,\, 0,\, 0)$ has components $T^{00} = T^{0x} = T^{x0} = T^{xx} = L/(4\pi x^2)$. The star sits at the origin.

   b) Let $\vec{X}$ be the null vector that separates the events of emission and reception of the radiation. Show that $\vec{X} \rightarrow_{\mathcal{O}} (x,\, x,\, 0,\, 0)$ for radiation observed at the event $(x,\, x,\, 0,\, 0)$. Show that the stress-energy tensor of (a) has the frame-invariant form

$$\boldsymbol{T} = \frac{L}{4\pi} \frac{\vec{X} \otimes \vec{X}}{(\vec{U}_s \cdot \vec{X})^4} \ ,$$

   where $\vec{U}_s$ is the star's four-velocity, $\vec{U}_s \rightarrow_{\mathcal{O}} (1,\, 0,\, 0,\, 0)$.

   c) Let the Earth-bound observer $\bar{\mathcal{O}}$, traveling with speed v away from the star in the x direction, measure the same radiation, again with the star on the $\bar{x}$ axis. Let $\vec{X} \rightarrow_{\bar{\mathcal{O}}} (R,\, R,\, 0,\, 0)$ and find R as a function of x. Express $T^{\bar{0}\bar{x}}$ in terms of R. Explain why R and $T^{\bar{0}\bar{x}}$ depend as they do on v.

4. Schutz 4.25

   Maxwell's equations for the electric and magnetic fields in vacuum, $\mathbf{E}$ and $\mathbf{B}$, in three-vector notation are

$$\begin{aligned} \boldsymbol{\nabla} \times \mathbf{B} - \frac{\partial}{\partial t}\mathbf{E} &= 4\pi \mathbf{J} \ , \\ \boldsymbol{\nabla} \times \mathbf{E} + \frac{\partial}{\partial t}\mathbf{B} &= 0 \ , \\ \boldsymbol{\nabla} \cdot \mathbf{E} &= 4\pi\rho \ , \\ \boldsymbol{\nabla} \cdot \mathbf{B} &= 0 \ , \end{aligned}$$

   in units where $\mu_0 = \epsilon_0 = c = 1$. (Here ρ is the density of the electric charge and $\mathbf{J}$ the current density.)

   a) An *antisymmetric* $\binom{2}{0}$ tensor $\mathbf{F}$ can be defined on spacetime by the equations $F^{0i} = E^i$, $F^{xy} = B^z$, $F^{yz} = B^x$, $F^{zx} = B^y$. Find from this definition all other components $F^{\mu\nu}$ in this frame and write them down in matrix in a matrix.

b) A rotation by an angle θ about the z axis is one kind of Lorentz transformation, with the matrix

$$\Lambda^{\beta'}{}_{\alpha} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

Show that the new components of $\mathbf{F}$,

$$F^{\alpha'\beta'} = \Lambda^{\alpha'}{}_{\mu}\Lambda^{\beta'}{}_{\nu}F^{\mu\nu} ,$$

define new electric and magnetic three-vector components (by the rule given in (a)) that are just the same as the components of the old $\mathbf{E}$ and $\mathbf{B}$ in the rotated three-space. (This shows that a spatial rotation of $\mathbf{F}$ makes a spatial rotation of $\mathbf{E}$ and $\mathbf{B}$.)

c) Define the current four-vector $\vec{J}$ by $J^0 = \rho$, $J^i = (\mathbf{J})^i$, and show that two of the Maxwell's equations are just

$$F^{\mu\nu}{}_{,\nu} = 4\pi J^{\mu} .$$

d) Show that the other two of Maxwell's equations are

$$F_{\mu\nu,\lambda} + F_{\nu\lambda,\mu} + F_{\lambda\mu,\nu} = 0 .$$

Note that there are only four independent equations here. That is, choose one index value, say zero. Then the three other values (1,2,3) can be assigned to μ, ν, λ in *any* order, producing the same equation (up to an overall sign) each time. Try it and see: it follows from antisymmetry of $F_{\mu\nu}$.

e) We have now expressed Maxwell's equations in tensor form. Show that conservation of charge, $J^{\mu}{}_{,\mu} = 0$ is implied by part c) above. (Hint: use the antisymmetry of $F_{\mu\nu}$.)

f) The charge density in any frame is J^0. Therefore the total charge in spacetime is $Q = \int J^0 dxdydz$, where the integral extends over an entire hypersurface $t =$ constant. Defining $\tilde{dt} = \tilde{n}$, a unit normal for this hypersurface, show that

$$Q = \int J^{\alpha} n_{\alpha} dxdydz .$$

g) Use Gauss's law and the result in part c) to show that the total energy enclosed within any closed two-surface $\mathcal{S}$ in the hypersurface $t =$ constant can be determined by doing an integral over $\mathcal{S}$ itself:

$$Q = \oint_{\mathcal{S}} F^{0i} n_i d\mathcal{S} = \oint \mathbf{E} \cdot \mathbf{n} d\mathcal{S} ,$$

where $\mathbf{n}$ is the unit normal to $\mathcal{S}$ in the hypersurface (*not* the same as $\tilde{n}$ in part f) above).

h) Perform a Lorentz transformation on $F^{\mu\nu}$ to a frame $\bar{\mathcal{O}}$ moving with velocity v in the x direction relative to the frame used in part a) above. In this frame define a three-vector $\bar{\mathbf{E}}$ with components $\bar{E}^i = F^{\bar{0}\bar{i}}$, and similarly for $\bar{\mathbf{B}}$ in analogy with a). In this way discover how $\mathbf{E}$ and $\mathbf{B}$ behave under a Lorentz transformation: they get mixed together! Thus, $\mathbf{E}$ and $\mathbf{B}$ themselves are not Lorentz invariant, but are merely components of $\mathbf{F}$, called the Faraday tensor, which is the invariant description of electromagnetic fields in relativity. If you think carefully, you will see that on physical grounds they *cannot* be invariant. In particular, the magnetic field is created by moving charges; but a charge moving in one frame may be at rest in another, so a magnetic field which exists in one frame may not exist in another. What is the same in *all* frames is the Faraday tensor: only its components get transformed.