PHSX 523: Homework Set 2

Due Date: Friday September 15, 2017, at the beginning of class.

Topic: Graduate General Relativity 1

1. Schutz 3.3

   a) Prove, by writing out all the terms, the validity of the following

   $$\tilde{p}(A^\alpha \vec{e}_\alpha) = A^\alpha \tilde{p}(\vec{e}_\alpha) \ .$$

   b) Let the componetns of $\tilde{p}$ be (-1,1,2,0), those of $\vec{A}$ be (2,1,0,-1), and those of $\vec{B}$ be (0,2,0,0). Find:

      i) $\tilde{p}(\vec{A})$

      ii) $\tilde{p}(\vec{B})$

      iii) $\tilde{p}(\vec{A} - 3\vec{B})$

      iv) $\tilde{p}(\vec{A}) - 3\tilde{p}(\vec{B})$

2. Schutz 3.10

   a) Given a frame $\mathcal{O}$ whose coordinates are x^α, show that

   $$\partial x^\alpha / \partial x^\beta = \delta^\alpha{}_\beta \ .$$

   b) For any two frames, we have

   $$\partial x^\beta / \partial x^{\bar{\alpha}} = \Lambda^\beta{}_{\bar{\alpha}} \ .$$

   Show that part a) and the chain rule imply

   $$\Lambda^\beta{}_{\bar{\alpha}} \Lambda^{\bar{\alpha}}{}_\mu = \delta^\beta{}_\mu \ .$$

   This is the inverse property again.

3. Schutz 3.14

   Let $\tilde{p} \to_{\mathcal{O}} (1,1,0,0)$ and $\tilde{q} \to_{\mathcal{O}} (-1,0,1,0)$ be two one-froms. Prove, by trying two vectors $\vec{A}$ and $\vec{B}$ as arguments, that $\tilde{p} \otimes \tilde{q} \neq \tilde{q} \otimes \tilde{p}$. Then find the components of $\tilde{p} \otimes \tilde{q}$.

4. Schutz 3.16

   a) Prove that $\mathbf{h}_{(\mathrm{s})}$ defined by

   $$\mathbf{h}_{(\mathrm{s})}(\vec{A}, \vec{B}) = \frac{1}{2}\mathbf{h}(\vec{A}, \vec{B}) + \frac{1}{2}\mathbf{h}(\vec{B}, \vec{A}) \ ,$$

   is a symmetric tensor.

   b) Prove that $\mathbf{h}_{(\mathrm{A})}$ defined by

   $$\mathbf{h}_{(\mathrm{A})}(\vec{A}, \vec{B}) = \frac{1}{2}\mathbf{h}(\vec{A}, \vec{B}) - \frac{1}{2}\mathbf{h}(\vec{B}, \vec{A}) \ ,$$

   is an antisymmetric tensor.

   c) Find the components of the symmetric parts of $\tilde{p} \otimes \tilde{q}$ defined in exercise 14, (your third problem above).

   d) Prove that if $\mathbf{h}$ is an antisymmetric $\binom{0}{2}$ tensor,

   $$\mathbf{h}(\vec{A}, \vec{A}) = 0 \ , \tag{1}$$

   for any vector $\vec{A}$.

5. Schutz 3.18

a) Find the one-forms mapped by the metric tensor $(\eta_{\mu\nu})$ from the vectors

$$\begin{aligned}
\vec{A} &\to_{\mathcal{O}} (1, 0, -1, 0) \ , \\
\vec{B} &\to_{\mathcal{O}} (0, 1, 1, 0) \ , \\
\vec{C} &\to_{\mathcal{O}} (-1, 0, -1, 0) \ , \\
\vec{D} &\to_{\mathcal{O}} (0, 0, 1, 1) \ ,
\end{aligned}$$

b) Find the vectors mapped from the inverse of the metric $(\eta^{\mu\nu})$ from the one-forms

$$\begin{aligned}
\tilde{p} &\to_{\mathcal{O}} (3, 0, -1, -1) \ , \\
\tilde{q} &\to_{\mathcal{O}} (1, -1, 1, 1) \ , \\
\tilde{r} &\to_{\mathcal{O}} (0, -5, -1, 0) \ , \\
\tilde{s} &\to_{\mathcal{O}} (-2, 1, 0, 0) \ ,
\end{aligned}$$