PHYS 515: GR 1 Final (Take-Home) Exam

Time: You will have 24 hours to complete the final and it must be turned in *only* through the my.physics website by Wednesday, May 13 at 1: 30 pm. Late submissions will not be possible.

Guidelines: You are allowed to make use of Carroll, your lecture notes, and homework. You will not be allowed to make use of computer software (including Mathematica and Maple), the internet, other books, other solutions, or each other. All work will be done by hand and must be shown in order to get credit. Take the time to write neatly and clearly and explain your steps to avoid ambiguity.

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1. [30 points] The gravitational analog of *bremsstrahlung* radiation is produced when two masses scatter off each other. Consider what happens when a small mass m scatters off a large mass M with impact parameter b and total energy $E = 0$. Take $M \gg m$ and $M/b \ll 1$. The motion of the small mass can be described by Newtonian physics, since $M/b \ll 1$.

   a) If the orbit lies in the (x, y) plane and if the large mass sits at $(x, y, z) = (0, 0, 0)$, calculate the gravitational wave amplitude for both polarizations at (x, y, z) = (0, 0, R).

   b) Since the motion is not periodic, the gravitational waves will be burst-like and composed of many different frequencies. On physical grounds, what do you expect the dominant frequency to be?

   c) Estimate the total energy radiated by the system.

   d) How does this compare to the peak kinetic energy of the small mass?

   Hint: The solution for the orbit can be found in Goldstein (2002). The solution is:

$$r = \frac{2b}{1 + \cos\theta} , \qquad t = \sqrt{\frac{2b^3}{M}} \left(\tan\frac{\theta}{2} + \frac{1}{3}\tan^3\frac{\theta}{2} \right) , \tag{1}$$

   with $t \in (-\infty, \infty)$. Rather than using the above implicit relation for $\theta(t)$ you might want to use

$$\dot{\theta} = \sqrt{\frac{M}{8b^3}} (1 + \cos\theta)^2 .$$

2. [40 points] In Boyer-Lindquist coordinates, the spacetime for a Kerr black hole has the line element

$$ds^2 = -\left(1 - \frac{2\beta r}{\Sigma}\right) dt^2 - \left(4\beta a r \frac{\sin^2\theta}{\Sigma}\right) dt d\phi + \frac{\Sigma}{\Delta} dr^2 + \Sigma\, d\theta^2 + \left(r^2 + a^2 + 2\beta a^2 r \frac{\sin^2\theta}{\Sigma}\right) \sin^2\theta\, d\phi^2 ,$$

   with

$$\Delta = r^2 - 2\beta r + a^2 , \qquad \Sigma = r^2 + a^2\cos^2\theta ,$$

   where β is a constant and a is another constant related to the black hole angular momentum.

   a) Find the relation between the constant β and the mass of the black hole.

   b) Find the Killing vectors of this spacetime.

   c) Where is the curvature singularity of this spacetime? What is its shape?

   d) Derive Kepler's third law ($\Omega^2 = M/r^3$ in Newtonian gravity) for an orbit on the equatorial plane.

3. [30 points] Consider a planet (with mass M_2) orbiting in a non-relativistic circular orbit around a star (with mass M_1), initially separated by a distance D_0.

   a) Compute $D(t)$, i.e., the change of their separation over time, due to the emission of gravitational waves.

   b) Consider the values for M_1 and M_2 corresponding to the Sun (~ 1.48 km) and the Earth, and take D_0 equal to the average current separation between the Earth and Sun ($\sim 1.799 \times 10^7$km), to evaluate how much D changes in 1 year.

   c) Based on the previous two results, what type of astrophysical systems would lead to the largest decay of their orbit?