Physics 509 Homework 1

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Spring 2023

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1 Index Gymnastics and Einstein Convention

Note that no distinction is made between raised and lowered indices in the problem.

(i) See (ii) with $\mathbf{y} \to \mathbf{x}$.

(ii) $\mathbf{x} \cdot \mathbf{y} = (x^\mu \hat{\mathbf{e}}_\mu)(y^\nu \hat{\mathbf{e}}_\nu) = x^\mu y^\nu \underbrace{\hat{\mathbf{e}}_\mu \hat{\mathbf{e}}_\nu}_{=\delta_{\mu\nu}} = x^\mu y_\mu$.

(iii) Note that $\delta_{\mu\nu}\delta_{\nu\rho} = 1$ if and only if $\mu = \rho$. This agrees with $\delta_{\mu\rho}$ for all values of μ and ρ, hence $\delta_{\mu\nu}\delta_{\nu\rho} = \delta_{\mu\rho}$.

(iv) $a_\mu = a_\nu(\hat{\mathbf{e}}_\nu \cdot \hat{\mathbf{e}}_\mu) = a_\nu \delta_{\mu\nu}$.

(v) $\delta_{\mu\mu} = \sum_{i=1}^{3} 1 = 3$.

(a) Writing out both sides of the expression explicitly, one finds on the LHS

$$(A^\mu B_\mu)(C^\nu D_\nu) = (a_1 b_1 + a_2 b_2 + a_3 b_3)(c_1 d_1 + c_2 d_2 + c_3 d_3)$$
$$= a_1 b_1 c_1 d_1 + a_1 b_1 c_2 d_2 + a_1 b_1 c_3 d_3 + a_2 b_2 c_1 d_1 + a_2 b_2 c_2 d_2 + a_2 b_2 c_3 d_3$$
$$+ a_3 b_3 c_1 d_1 + a_3 b_3 c_2 d_2 + a_3 b_3 c_3 d_3,$$

whereas on the RHS,

$$(A^\mu C^\nu)(B_\mu D_\nu)$$
$$= a_1 c_1 b_1 d_1 + a_1 c_2 b_1 d_2 + a_1 c_3 b_1 d_3 + a_2 c_1 b_2 d_1 + a_2 c_2 b_2 d_2 + a_2 c_3 b_2 d_3$$
$$+ a_3 c_1 b_3 d_1 + a_3 c_2 b_3 d_2 + a_3 c_3 b_3 d_3.$$

These expressions are clearly equal.

(b) Suppose $A_{\mu\nu} = -A_{\mu\nu}$ and $B^{\mu\nu} = B^{\nu\mu}$. Then writing out terms explicitly yields

$$A_{\mu\nu}B^{\mu\nu} = a_{11}b_{11} + a_{12}b_{12} + a_{13}b_{13} + a_{21}b_{21} + a_{22}b_{22} + a_{23}b_{23} + a_{31}b_{31} + a_{32}b_{32} + a_{33}b_{33}.$$

Canceling all the diagonal terms since $a_{ii} = -a_{ii} \implies a_{ii} = 0$ for any i and replacing $a_{ji} = -a_{ij}$ and $b_{ji} = b_{ij}$ whenever $i < j$ yields

$$A_{\mu\nu} = a_{12}b_{12} + a_{13}b_{13} + a_{23}b_{23} - a_{12}b_{12} - a_{13}b_{13} - a_{23}b_{23} = 0.$$

This can be shown more concisely by relabeling indices:

$$A_{\mu\nu}B^{\mu\nu} \xrightarrow{\mu\leftrightarrow\nu} A_{\nu\mu}B^{\nu\mu} = -A_{\mu\nu}B^{\mu\nu} \implies A_{\mu\nu}B^{\mu\nu} = 0.$$

The last equality follows since simply relabelling indices should not change the result.

2 Antisymmetry

(a) It suffices to show that 123, 231, and 312 are all even permutations of 123:

$$123 \xrightarrow{\text{id}} 123$$

$$123 \xrightarrow{(12)} 213 \xrightarrow{(23)} 231$$

$$123 \xrightarrow{(23)} 132 \xrightarrow{(12)} 312.$$

Each of these contain an even number of transpositions and are therefore even permutations: $\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$.

In contrast,

$$1234 \xrightarrow{(12)} 2134 \xrightarrow{(23)} 2314 \xrightarrow{(34)} 2341$$

contains an odd number of transpositions; therefore, $\epsilon_{1234} = -\epsilon_{2341}$.

(b) First, note that only the relative permutations between unprimed and primed indices matters since

$$\epsilon_{ijk}\epsilon_{i'j'k'} = \text{sgn}(\sigma)\,\text{sgn}(\tau) = \text{sgn}(\sigma\tau),$$

where σ and τ act on unprimed and primed indices respectively. Without loss of generality let τ denote a permutation of the primed indices relative to ijk. Then

$$\epsilon_{ijk}\epsilon_{i'j'k'} = \sum_{\tau\in S_3} \text{sgn}(\tau)\delta_{i\tau(i')}\delta_{j\tau(j')}\delta_{k\tau(k')}$$

$$= \delta_{ii'}\delta_{jj'}\delta_{kk'} - \delta_{ij'}\delta_{ji'}\delta_{kk'} + \delta_{ij'}\delta_{jk'}\delta_{ki'} - \delta_{ik'}\delta_{jj'}\delta_{ki'}$$

$$+ \delta_{ik'}\delta_{ji'}\delta_{kj'} - \delta_{ii'}\delta_{jk'}\delta_{kj'}.$$

(c) Setting $i = i'$ in part (b) yields

$$\epsilon_{ijk}\epsilon_{ij'k'} = \delta_{jj'}\delta_{kk'} - \delta_{jk'}\delta_{kj'}. \tag{1}$$

(d) The result from (b) clearly generalizes to

$$\epsilon_{ijk\ell}\epsilon_{i'j'k'\ell'} = \sum_{\tau \in S_4} \mathrm{sgn}(\tau)\delta_{i\tau(i')}\delta_{j\tau(j')}\delta_{k\tau(k')}\delta_{\ell\tau(\ell')}.$$

Setting $i = i'$, we recover a similar result to that of part (b) (as expected):

$$\epsilon_{ijk\ell}\epsilon_{i'j'k'\ell'} = \delta_{ii'}\delta_{jj'}\delta_{kk'} - \delta_{ij'}\delta_{ji'}\delta_{kk'} + \delta_{ij'}\delta_{jk'}\delta_{ki'} - \delta_{ik'}\delta_{jj'}\delta_{ki'} + \delta_{ik'}\delta_{ji'}\delta_{kj'} - \delta_{ii'}\delta_{jk'}\delta_{kj'}.$$

3   Vector Products

(i) By definition, $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = a_k(\epsilon_{ijk}b_ic_j) = \epsilon_{ijk}a_kb_ic_j$. This is clearly invariant under any even permutation of indices which shows that

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}).$$

(ii) Plugging in the definition and using our earlier results,

$$\begin{aligned}
\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= \mathbf{a} \times (\epsilon_{ijk}b_jc_k\hat{\mathbf{e}}_i) \\
&= \epsilon_{ij'k'}\epsilon_{ijk}a_{k'}b_jc_k\hat{\mathbf{e}}_{j'} \\
&= (\delta_{jj'}\delta_{kk'} - \delta_{jk'}\delta_{kj'})a_{k'}b_jc_k\hat{\mathbf{e}}_{j'} \qquad &\text{(by (1))} \\
&= a_kc_kb_j\hat{\mathbf{e}}_j - a_jb_jc_k\hat{\mathbf{e}}_k.
\end{aligned}$$

Expressing this in vector notation yields

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}). \qquad (2)$$

One should also be aware that the cross product is one of the few elementary operations which is **not** associative, so that parenthesis are typically required to disambiguate expressions containing multiple cross products; e.g.

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \neq \mathbf{a} \times (\mathbf{b} \times \mathbf{c}).$$

(iii) Again, using index notation and previous identities,

$$\begin{aligned}
(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) &= (\epsilon_{ijk}a_jb_k\hat{\mathbf{e}}_i) \cdot (\epsilon_{i'j'k'}c_{j'}d_{k'}\hat{\mathbf{e}}_{i'}) \\
&= (\epsilon_{ijk}a_jb_k)(\epsilon_{ij'k'}c_{j'}d_{k'}) \qquad &(\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_{i'} = \delta_{ii'}) \\
&= (\delta_{jj'}\delta_{kk'} - \delta_{jk'}\delta_{kj'})a_jb_kc_{j'}d_{k'} \qquad &\text{(by (1))} \\
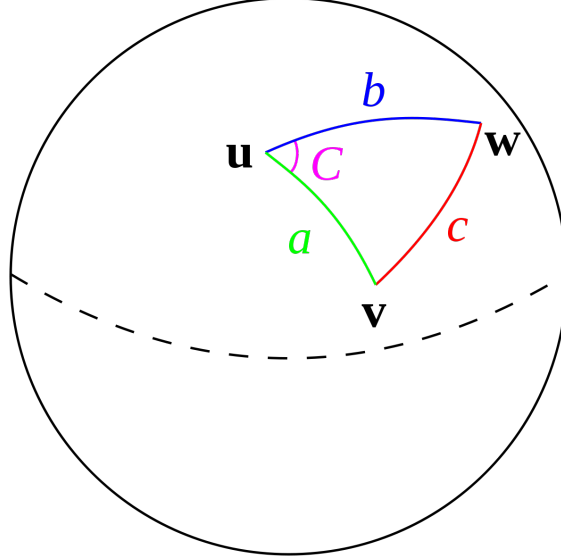&= a_jb_kc_jd_k - a_jb_kc_kd_j.
\end{aligned}$$

Figure 1: Definitions of arc lengths, angles, and vectors used in the derivation of the law of spherical cosines. Note that the vectors $\mathbf{u}$, $\mathbf{w}$, and $\mathbf{v}$ are placed at the origin (center of the sphere) and are of unit length.

Expressing this in vector notation yields

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}). \tag{3}$$

(iv) First consider the spherical law of cosines,

$$\cos(c) = \cos(a)\cos(b) + \sin(a)\sin(b)\cos(C), \tag{4}$$

where the arc lengths (angles between the unit vectors), a, b, c, and angle C are shown in figure 1. Using the familiar identities,

$$\mathbf{a} \cdot \mathbf{b} = ab\cos(\theta) \quad \text{and} \quad |\mathbf{a} \times \mathbf{b}| = ab\sin(\theta),$$

and making the formal substitutions $\mathbf{a} \to \mathbf{u}$, $\mathbf{b} \to \mathbf{v}$, $\mathbf{c} \to \mathbf{u}$, and $\mathbf{d} \to \mathbf{w}$ (so our notation is consistent with that in figure 1), the LHS of equation (3) can be written as

$$(\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{u} \times \mathbf{w}) = \sin(a)\sin(b)\cos(C).$$

The RHS of equation (3) yields

$$(\mathbf{u} \cdot \mathbf{u})(\mathbf{v} \cdot \mathbf{w}) - (\mathbf{u} \cdot \mathbf{v})(\mathbf{u} \cdot \mathbf{w}) = \cos(c) - \cos(a)\cos(b),$$

which rearranges to the desired result (equation (4)).

The spherical law of sines is

$$\frac{\sin(A)}{\sin(a)} = \frac{\sin(B)}{\sin(b)} = \frac{\sin(C)}{\sin(c)},$$ (5)

where a, b, and c are the arcs on the surface of the sphere (equivalently, the corresponding angles between $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ since the sphere is of unit radius) and A, B, and C are the spherical angles opposite their respective arcs (e.g., the relationship between c and C is depicted in figure 1).

Following the hint provided, we first prove the identity:

$$\mathbf{a} \cdot [(\mathbf{a} \times \mathbf{b}) \times (\mathbf{a} \times \mathbf{c})] = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}).$$ (6)

Plugging in the definition of the cross-product,

$$
\begin{aligned}
\mathbf{a} \cdot [(\mathbf{a} \times \mathbf{b}) \times (\mathbf{a} \times \mathbf{c})] &= \mathbf{a} \cdot \left[(\epsilon_{ijk} a_j b_k \hat{\mathbf{e}}_i) \times (\epsilon_{i'j'k'} a_{j'} b_{k'} \hat{\mathbf{e}}_{i'}) \right] \\
&= \mathbf{a} \cdot \left[\epsilon_{\ell i i'} \epsilon_{ijk} \epsilon_{i'j'k'} a_j b_k a_{j'} b_{k'} \hat{\mathbf{e}}_\ell \right] \\
&= \epsilon_{\ell i i'} \epsilon_{ijk} \epsilon_{i'j'k'} a_\ell a_j b_k a_{j'} b_{k'} \\
&= (\delta_{i'j} \delta_{\ell k} - \delta_{i'k} \delta_{\ell j}) \epsilon_{i'j'k'} a_\ell a_j b_k a_{j'} b_{k'} \qquad \text{(by (1))} \\
&= \underbrace{\epsilon_{jj'k'} a_k a_j a_{j'} b_k c_{k'}}_{\propto (\mathbf{a} \times \mathbf{a}) \cdot \mathbf{c} = 0} \underbrace{-\epsilon_{kj'k'} a_\ell a_\ell a_{j'} b_k c_{k'}}_{=\epsilon_{j'kk'} a_{j'} b_k c_{k'}} \\
&= a_i (\epsilon_{ijk} b_j c_k) \qquad\qquad\qquad \text{(relabelling indices)}.
\end{aligned}
$$

Note, in the second to last line, $a_\ell a_\ell = 1$ since these are unit vectors. This establishes identity (6).

We can also just plug into the identity established in part (ii),

$$
\begin{aligned}
\mathbf{a} \cdot [(\mathbf{a} \times \mathbf{b}) \times (\mathbf{a} \times \mathbf{c})] &= \mathbf{a} \cdot \left[((\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}) \mathbf{a} - ((\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a}) \mathbf{c} \right] \\
&= (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c},
\end{aligned}
$$

where in going to the last line we have used that $\mathbf{a} \cdot \mathbf{a} = 1$ and $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = 0$.

Using this formula with $\mathbf{a} \to \mathbf{u}$, $\mathbf{b} \to \mathbf{v}$, and $\mathbf{c} \to \mathbf{w}$ yields

$$\mathbf{u} \cdot [(\mathbf{u} \times \mathbf{v}) \times (\mathbf{u} \times \mathbf{w})] = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}).$$ (7)

Since we are in three dimensions with $\mathbf{u}$ perpendicular to both $(\mathbf{u} \times \mathbf{v})$ and $(\mathbf{u} \times \mathbf{w})$, then $\mathbf{u}$ is parallel to the cross product $(\mathbf{u} \times \mathbf{v}) \times (\mathbf{u} \times \mathbf{w})$ and hence we have no cosine term in the product on the LHS. The RHS of (7) is invariant under even permutations of vectors (from

part (i)). Hence, we can write

$$|(\mathbf{u} \times \mathbf{v}) \times (\mathbf{u} \times \mathbf{w})| = |(\mathbf{v} \times \mathbf{w}) \times (\mathbf{v} \times \mathbf{u})| = |(\mathbf{w} \times \mathbf{u}) \times (\mathbf{w} \times \mathbf{v})|$$
$$\implies \sin(a)\sin(b)\sin(C) = \sin(a)\sin(c)\sin(B) = \sin(b)\sin(c)\sin(A),$$

which reduces to the desired result, equation (5). Note that the angle between $(\mathbf{u} \times \mathbf{v})$ and $(\mathbf{u} \times \mathbf{w})$, for example, is just C.

4   Bernoulli and Vector Products

Let's first rewrite the expression $\mathbf{u} \times (\boldsymbol{\nabla} \times \mathbf{v})$ in index notation.

$$
\begin{aligned}
\mathbf{u} \times (\boldsymbol{\nabla} \times \mathbf{v}) &= \mathbf{u} \times (\epsilon_{ijk}(\partial_i v_j)\hat{\mathbf{e}}_k) \\
&= \epsilon_{i'j'k'} u_{i'} \left(\epsilon_{ijk}(\partial_i v_j)\hat{\mathbf{e}}_k\right)_{j'} \hat{\mathbf{e}}_{k'} \\
&= \epsilon_{i'kk'}\epsilon_{ijk} u_{i'}(\partial_i v_j)\hat{\mathbf{e}}_{k'} \\
&= -\epsilon_{ki'k'}\epsilon_{kij} u_{i'}(\partial_i v_j)\hat{\mathbf{e}}_{k'} & \text{(re-order indices)} \\
&= -(\delta_{ii'}\delta_{jk'} - \delta_{ik'}\delta_{jk'}) u_{i'}(\partial_i v_j)\hat{\mathbf{e}}_{k'} & \text{(by (1))} \\
&= u_j(\partial_i v_j)\hat{\mathbf{e}}_i - u_i(\partial_i v_j)\hat{\mathbf{e}}_j.
\end{aligned}
$$

This expression is sometimes written using Feynman's subscript notation,

$$\mathbf{u} \times (\boldsymbol{\nabla} \times \mathbf{v}) = \boldsymbol{\nabla}_{\mathbf{v}}(\mathbf{u} \cdot \mathbf{v}) - (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{v},$$

where $\boldsymbol{\nabla}_{\mathbf{v}}$ acts only on the $\mathbf{v}$ coordinates to the right. Using $\frac{1}{2}\boldsymbol{\nabla}\mathbf{v}^2 = v_i(\partial_j v_i)\hat{\mathbf{e}}_j$, we can write

$$\mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{v}) = \frac{1}{2}\boldsymbol{\nabla}\mathbf{v}^2 - (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v}. \tag{8}$$

Using this identity, Euler's equation for fluid motion,

$$\dot{\mathbf{v}} + (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v} = -\boldsymbol{\nabla}h$$

becomes
$$\dot{\mathbf{v}} - \mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{v}) + \frac{1}{2}\boldsymbol{\nabla}\mathbf{v}^2 = -\boldsymbol{\nabla}h \implies \dot{\mathbf{v}} - \mathbf{v} \times \boldsymbol{\omega} = -\boldsymbol{\nabla}\left(\frac{1}{2}\mathbf{v}^2 + h\right),$$

where the final expression has been written in terms of the vorticity, $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{v}$.

For steady flow ($\dot{\mathbf{v}} = \mathbf{0}$), the quantity $\frac{1}{2}\mathbf{v}^2 + h$ is constant along streamlines since

$$-\mathbf{v} \cdot \nabla\left(\frac{1}{2}\mathbf{v}^2 + h\right) = \mathbf{v} \cdot (\mathbf{v} \times \boldsymbol{\omega}) = 0.$$

5 Antisymmetry and Determinants

(a) Given the definition of the determinant,

$$\det(\mathbf{A}) = \epsilon_{j_1 j_2 \ldots j_n} A_{1j_1} A_{2j_2} \ldots A_{nj_n}, \tag{9}$$

relabel the indices by a permutation; i.e., by σ such that $\sigma(k) = i_k$.

$$\det(\mathbf{A}) = \epsilon_{j_{\sigma(1)} j_{\sigma(2)} \ldots j_{\sigma(n)}} A_{\sigma(1) j_{\sigma(1)}} A_{\sigma(2) j_{\sigma(2)}} \ldots A_{\sigma(n) j_{\sigma(n)}}$$

$$= \epsilon_{j_{i_1} j_{i_2} \ldots j_{i_n}} A_{i_1 j_{i_1}} A_{i_2 j_{i_2}} \ldots A_{i_n j_{i_n}}$$

$$= \epsilon_{i_1 i_2 \ldots i_n} \epsilon_{j_1 j_2 \ldots j_n} A_{i_1 j_1} A_{i_2 j_2} \ldots A_{i_n j_n}.$$

In the last line, the double subscripts, j_{i_k} terms, have been relabelled to j_k terms. This leaves the product of matrix elements unchanged while introducing a factor of $\epsilon_{i_1 i_2 \ldots i_n}$ from reordering the $\epsilon_{j_{i_1} j_{i_2} \ldots j_{i_n}}$ term. This establishes the desired result,

$$\epsilon_{i_1 i_2 \ldots i_n} \det(\mathbf{A}) = \epsilon_{j_1 j_2 \ldots j_n} A_{i_1 j_1} A_{i_2 j_2} \ldots A_{i_n j_n}. \tag{10}$$

This result can be used to show the Cauchy-Binet formula, $\det(\mathbf{AB}) = \det(\mathbf{A}) \det(\mathbf{B})$.

$$\det(\mathbf{AB}) = \epsilon_{j_1 j_2 \ldots j_n} A_{1 k_1} B_{k_1 j_1} A_{2 k_2} B_{k_2 j_2} \ldots A_{n k_n} B_{k_n j_n}$$

$$= A_{1 k_1} A_{2 k_2} \ldots A_{n k_n} \underbrace{(\epsilon_{j_1 j_2 \ldots j_n} B_{k_1 j_1} B_{k_2 j_2} \ldots B_{k_n j_n})}_{=\epsilon_{k_1 k_2 \ldots k_n} \det(\mathbf{B})} \qquad \text{(by (10))}$$

$$= \underbrace{\epsilon_{k_1 k_2 \ldots k_n} A_{1 k_1} A_{2 k_2} \ldots A_{n k_n}}_{=\det(\mathbf{A})} \det(\mathbf{B}) \qquad \text{(by (9))}$$

$$= \det(\mathbf{A}) \det(\mathbf{B}).$$

(b) We now repeat the above exercise but using the language of differential forms.

(i) Since V is n dimensional, $\{\omega \,|\, \omega : V^n \to \mathbb{C}\}$ forms a one-dimensional vector space over $\mathbb{C}$. Hence, there is only one form up to multiplicative constant. One should check for themselves that the axioms for a vector space are indeed satisfied by the space of forms. We will choose this constant in what follows by the action of the form on the standard basis, $\{\hat{\mathbf{e}}_k\}$, by demanding that $\omega(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \ldots, \hat{\mathbf{e}}_n) = 1$.

More directly, you can also use the skew-symmetric and n-linearity to write any such form as a determinant times its evaluation on the standard basis (see below). Since the evaluation of the form on the standard basis is unity, this uniquely determines the form up to a multiplicative constant.

(ii) Now we want to show $\{\mathbf{x}_k\}_{k=1}^n$ are linearly independent if and only if $\omega(\mathbf{x}_1, \ldots, \mathbf{x}_n) \neq 0$. Or equivalently, $\{\mathbf{x}_k\}_{k=1}^n$ linearly dependent if and only if $\omega(\mathbf{x}_1, \ldots, \mathbf{x}_n) = 0$ (this is just the contrapositive). For convenience of notation, write $\mathbf{x}_1$ as $\mathbf{x}^{(1)}$.

($\Longrightarrow$) First, suppose $\omega(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(n)}) = 0$. The goal here is to reduce the evaluation of the form on the vectors $\{\mathbf{x}^k\}$ to an expression containing the evaluation of the form on only the standard basis, which we have already specified. Define the matrix

$$\mathbf{X} = (\mathbf{x}^{(1)} \ \mathbf{x}^{(2)} \ \ldots \ \mathbf{x}^{(n)}).$$

Then,

$$
\begin{aligned}
\omega(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(n)}) &= \omega(x_{k_1}^{(1)} \hat{\mathbf{e}}_{k_1}, x_{k_2}^{(2)} \hat{\mathbf{e}}_{k_2}, \ldots, x_{k_n}^{(n)} \hat{\mathbf{e}}_{k_n}) \\
&= x_{k_1}^{(1)} x_{k_2}^{(2)} \ldots x_{k_n}^{(n)} \omega(\hat{\mathbf{e}}_{k_1}, \hat{\mathbf{e}}_{k_2}, \ldots, \hat{\mathbf{e}}_{k_n}) \\
&= \underbrace{\epsilon_{k_1 k_2 \ldots k_n} x_{k_1}^{(1)} x_{k_2}^{(2)} \ldots x_{k_n}^{(n)}}_{=\det(\mathbf{X})} \underbrace{\omega(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \ldots, \hat{\mathbf{e}}_n)}_{=1}.
\end{aligned}
$$

This shows that if $\omega(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(n)}) = 0$ then $\det(\mathbf{X}) = 0$ which implies that $\{\mathbf{x}_k\}_{k=1}^n$ are linearly dependent.

($\Longleftarrow$) Conversely, if $\{\mathbf{x}_k\}_{k=1}^n$ are linearly dependent then, without loss of generality, we can write $\mathbf{x}_1 = \sum_{k=2}^n c_k \mathbf{x}_k$ for some coefficients c_k. Then

$$\omega(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) = \sum_{k=2}^n c_k \omega(\mathbf{x}_k, \mathbf{x}_2, \ldots, \mathbf{x}_n) = 0.$$

Every term in the sum is zero since the (antisymmetric) form contains repeated elements; hence, the sum is identically zero.

Now define the determinant of the linear map $A : V \to V$ by

$$(\det \mathbf{A}) \, \omega(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) = \omega(\mathbf{A}\mathbf{x}_1, \mathbf{A}\mathbf{x}_2, \ldots, \mathbf{A}\mathbf{x}_n). \tag{11}$$

Writing everything in terms of the standard basis, $\mathbf{x}^{(k)} = x_i^{(k)} \hat{\mathbf{e}}_i$, $\mathbf{A}\mathbf{x}^{(k)} = A_{ij} x_j^{(k)} \hat{\mathbf{e}}_i$, and using $\omega(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \ldots, \hat{\mathbf{e}}_n) = 1$, one finds

$$
\begin{aligned}
\omega(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(n)}) &= \omega(x_{j_1}^{(1)} \hat{\mathbf{e}}_{j_2}, x_{j_2}^{(2)} \hat{\mathbf{e}}_{j_2}, \ldots, x_{j_n}^{(n)} \hat{\mathbf{e}}_{j_n}) \\
&= x_{j_1}^{(1)} x_{j_2}^{(2)} \ldots x_{j_n}^{(n)} \omega(\hat{\mathbf{e}}_{j_2}, \hat{\mathbf{e}}_{j_2}, \ldots, \hat{\mathbf{e}}_{j_n}) \qquad (\omega \text{ is multilinear}) \\
&= \epsilon_{j_1 j_2 \ldots j_n} x_{j_1}^{(1)} x_{j_2}^{(2)} \ldots x_{j_n}^{(n)} \underbrace{\omega(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \ldots, \hat{\mathbf{e}}_n)}_{=1} \qquad (\text{by skew-symmetry}).
\end{aligned}
$$

8

Similarly,

$$
\begin{aligned}
\omega(\mathbf{A}\mathbf{x}_1, \mathbf{A}\mathbf{x}_2, \ldots, \mathbf{A}\mathbf{x}_n) &= \omega(A_{i_1 j_1} x_{j_1}^{(1)} \hat{\mathbf{e}}_{i_1}, A_{i_2 j_2} x_{j_2}^{(2)} \hat{\mathbf{e}}_{i_2}, \ldots, A_{i_n j_n} x_{j_n}^{(n)} \hat{\mathbf{e}}_{i_n}) \\
&= A_{i_1 j_1} x_{j_1}^{(1)} A_{i_2 j_2} x_{j_2}^{(2)} \ldots A_{i_n j_n} x_{j_n}^{(n)} \omega(\hat{\mathbf{e}}_{i_1}, \hat{\mathbf{e}}_{i_2}, \ldots, \hat{\mathbf{e}}_{i_n}) \\
&= \epsilon_{i_1 i_2 \ldots i_n} A_{i_1 j_1} A_{i_2 j_2} \ldots A_{i_n j_n} x_{j_1}^{(1)} x_{j_2}^{(2)} \ldots x_{j_n}^{(n)} \underbrace{\omega(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \ldots, \hat{\mathbf{e}}_n)}_{=1}.
\end{aligned}
$$

Using these expressions in equation (11) reduces to

$$
\det(\mathbf{A}) = \epsilon_{i_1 i_2 \ldots i_n} \epsilon_{j_1 j_2 \ldots j_n} A_{i_1 j_1} A_{i_2 j_2} \ldots A_{i_n j_n},
$$

which agrees with equation (10).

The proof of the Cauchy-Binet formula is now trivial:

$$
\begin{aligned}
(\det \mathbf{A})(\det \mathbf{B})\omega(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) &= (\det \mathbf{A})\omega(\mathbf{B}\mathbf{x}_1, \mathbf{B}\mathbf{x}_2, \ldots, \mathbf{B}\mathbf{x}_n) \\
&= \omega(\mathbf{A}\mathbf{B}\mathbf{x}_1, \mathbf{A}\mathbf{B}\mathbf{x}_2, \ldots, \mathbf{A}\mathbf{B}\mathbf{x}_n) \\
&= (\det \mathbf{A}\mathbf{B})\omega(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n).
\end{aligned}
$$