1) Old Exam problem: A two-form is expressed in Cartesian coordinates as,

$$\omega = \frac{1}{r^3}(zdxdy + xdydz + ydzdx)$$

where $r = \sqrt{x^2 + y^2 + z^2}$.

a) Evaluate $d\omega$ for $r \neq 0$.

b) Evaluate the integral

$$\Phi = \int_P \omega$$

over the infinite plane $P = \{-\infty < x < \infty, -\infty < y < \infty, z = 1\}$.

c) A sphere is embedded into $\mathbb{R}^3$ by the map φ, which takes the point $(\theta, \phi) \in S^2$ to the point $(x, y, z) \in \mathbb{R}^3$, where

$$\begin{aligned} x &= R\cos\phi\sin\theta \\ y &= R\sin\phi\sin\theta \\ z &= R\cos\theta. \end{aligned}$$

Pull back ω and find the 2-form $\varphi^*\omega$ on the sphere. (**Hint**: The form $\varphi^*\omega$ is both familiar and simple. If you end up with an intractable mess of trigonometric functions, you have made an algebraic error.)

d) By exploiting the result of part c), or otherwise, evaluate the integral

$$\Phi = \int_{S^2(R)} \omega$$

where $S^2(R)$ is the surface of a two-sphere of radius R centered at the origin.

2) Sphere Area: The sphere S^n can be thought of as the locus of points in $\mathbb{R}^{n+1}$ obeying $\sum_{i=1}^{n+1}(x^i)^2 = 1$. Use its invariance under orthogonal transformations to show that the element of surface "volume" of the n-sphere can be written as

$$d(\text{Volume on } S^n) = \frac{1}{n!}\epsilon_{\alpha_1\alpha_2\ldots\alpha_{n+1}} x^{\alpha_1}\, dx^{\alpha_2} \ldots dx^{\alpha_{n+1}}.$$

Use Stokes' theorem to relate the integral of this form over the *surface* of the sphere to the volume of the *solid* unit sphere. Confirm that we get the correct proportionality between the volume of the solid unit sphere and the volume or area of its surface.

3) Push and Pull: Use a local co-ordinate system to work the following exercises:

a) Show that the operation of taking an exterior derivative commutes with a pull back:

$$d\left[\phi^*\omega\right] = \phi^*(d\omega), \quad \omega \in \bigwedge^p T^*N.$$

b) If the map $\phi : M \to N$ is invertible then we may push forward a vector field X on M to get a vector field $\phi_* X$ on N. Show that

$$\mathcal{L}_X[\phi^*\omega] = \phi^*\left[\mathcal{L}_{\phi_* X}\omega\right], \quad \omega \in \bigwedge^p T^*N.$$

c) Again assume that $\phi : M \to N$ is invertible. By using the co-ordinate expressions for the Lie bracket and the effect of a push-forward, show that if X, Y are vector fields on TM then

$$\phi_*([X, Y]) = [\phi_* X, \phi_* Y],$$

as vector fields on TN.

4) Sterographic Co-ordinates: The stereographic map $S^2 \to \mathbb{C}$ takes the point on S^2 with spherical polar co-ordinates θ, ϕ to the complex number

$$\zeta = e^{i\phi} \tan\theta/2.$$

We can therefore set $\zeta = \xi + i\eta$ and take ζ, η as defining a *stereographic co-ordinate system* on the sphere. Show that in these co-ordinates the sphere metric is given by

$$\begin{aligned} g(\ ,\ ) &\equiv d\theta \otimes d\theta + \sin^2\theta \, d\phi \otimes d\phi \\ &= \frac{2}{(1+|\zeta|^2)^2}(d\overline{\zeta} \otimes d\zeta + d\zeta \otimes d\overline{\zeta}) \\ &= \frac{4}{(1+\xi^2+\eta^2)^2}(d\xi \otimes d\xi + d\eta \otimes d\eta), \end{aligned}$$

and the area 2-form becomes

$$\begin{aligned} \Omega &\equiv \sin\theta \, d\theta \wedge d\phi \\ &= \frac{2i}{(1+|\zeta|^2)^2} \, d\zeta \wedge d\overline{\zeta} \\ &= \frac{4}{(1+\xi^2+\eta^2)^2} \, d\xi \wedge d\eta. \end{aligned} \tag{1}$$

5) Bogomolnyi Equations In this problem you will find the spin field $n : x \mapsto \mathbf{n}(x)$ that minimizes the non-linear σ-model energy functional

$$E[\mathbf{n}] = \frac{1}{2}\int_{\mathbb{R}^2} \left(|\nabla n^1|^2 + |\nabla n^2|^2 + |\nabla n^3|^2\right) \, dx^1 dx^2$$

for a given positive winding number N.

a) Use the results of the preceding exercise to write the winding number $N = \frac{1}{4\pi}\int n^*\Omega$, and the energy functional $E[\mathbf{n}]$ as

$$\begin{aligned}
4\pi N &= \int \frac{4}{(1+|\xi|^2+|\eta|^2)^2}\left(\partial_1\xi\partial_2\eta - \partial_1\eta\partial_2\xi\right)\,dx^1dx^2,\\
E[\mathbf{n}] &= \frac{1}{2}\int \frac{4}{(1+|\xi|^2+|\eta|^2)^2}\left((\partial_1\xi)^2+(\partial_2\xi)^2+(\partial_1\eta)^2+(\partial_2\eta)^2\right)\,dx^1dx^2,
\end{aligned}$$

where ξ and η are stereographic co-ordinates on S^2 specifying the direction of the unit vector $\mathbf{n}$.

b) Deduce the inequality

$$E - 4\pi N \equiv \frac{1}{2}\int \frac{4}{(1+|\xi|^2+|\eta|^2)^2}|(\partial_1 + i\partial_2)(\xi+i\eta)|^2\,dx^1dx^2 > 0.$$

c) Deduce that for winding number $N > 0$, the minimum energy solutions have energy $E = 4\pi N$ and are obtained by solving the first-order linear equation

$$\left(\frac{\partial}{\partial x^1} + i\frac{\partial}{\partial x^2}\right)(\xi + i\eta) = 0.$$

d) Solve the equation in part c) and show that the minimal energy solutions with winding number $N > 0$ are given by

$$\xi + i\eta = \lambda\frac{(z-a_1)\ldots(z-a_N)}{(z-b_1)\ldots(z-b_N)}$$

where $z = x^1 + ix^2$, and $\lambda, a_1, \ldots, a_N$, and $b_1, \ldots, b_N$, are arbitrary (except for the condition that no a coincides with any b) complex numbers.

e) Repeat the analysis for $N < 0$. Show that the solutions are given in terms of rational functions of $\bar{z} = x^1 - ix^2$.

The idea of combining the energy functional and the topological charge into a single, manifestly positive, functional is due to Bogomolnyi. The the resulting first order linear equation is therefore called a *Bogomolnyi equation*. If we had tried to find a solution directly in terms of $\mathbf{n}$, we would have ended up with a horribly non-linear second-order partial differential equation..