Physics 508 | Mathematical Methods in Physics I | Prof. M. Stone

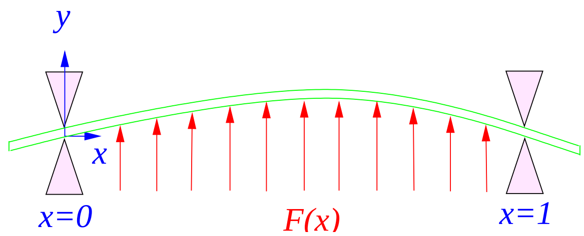
Handout 7 | | 2117 ESB

Fall 2018 | HOMEWORK 7 | University of Illinois

# Mathematical Methods in Physics I

## Homework 7

**1) Flexible rod again**: A flexible rod is supported near its ends by means of knife edges that constrain its position, but not its slope or curvature. It is acted on by by a force $F(x)$.




*Simply supported rod.*

The deflection of the rod is found by solving the the boundary value problem

$$\frac{d^4y}{dx^4} = F(x), \quad y(0) = y(1) = 0, \quad y''(0) = y''(1) = 0.$$

We wish to find the Green function $G(x, y)$ that facilitates the solution of this problem.

a) If the differential operator and domain (boundary conditions) above is denoted by $L$, what is the operator and domain for $L^\dagger$? Is the problem self-adjoint?

b) Are there any zero-modes? Does $F$ have to satisfy any conditions for the solution to exist?

c) Write down the conditions, if any, obeyed by $G(x, y)$ and its derivatives $\partial_x G(x, y)$, $\partial^2_{xx} G(x, y)$, $\partial^3_{xxx} G(x, y)$ at $x = 0$, $x = y$, and $x = 1$.

d) Using the conditions above, find $G(x, y)$. (This requires some boring algebra — but if you start from the "jump condition" and work down, it can be completed in under a page)

e) Is your Green function symmetric $(G(x, y) = G(y, x))$? Is this in accord with the self-adjointness or not of the problem? (You can use this as a check of your algebra.)

f) Write down the integral giving the general solution of the boundary value problem. Assume, if necessary, that $F(x)$ is in the range of the differential operator. Differentiate your answer and see if it does indeed satisfy the differential equation and boundary conditions.

**2) Hot ring**: The equation governing the steady state heat flow on thin ring of unit circumference is

$$-y'' = f, \quad 0 < x < 1, \quad y(0) = y(1), \quad y'(0) = y'(1).$$

a) This problem has a zero mode. Find the zero mode and the consequent condition on $f(x)$ for a solution to exist.

b) Verify that a suitable modified Green function for the problem is

$$g(x,y) = \frac{1}{2}(x-y)^2 - \frac{1}{2}|x-y|.$$

You will need to verify that $g(x,y)$ satisfies both the differential equation *and* the boundary conditions.

**3) Lattice Green Functions** . The $k \times k$ matrices

$$\mathbf{T}_1 = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & 0 & \dots & 0 \\ 0 & -1 & 2 & -1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & -1 & 2 & -1 & 0 \\ 0 & \dots & 0 & 0 & -1 & 2 & -1 \\ 0 & \dots & 0 & 0 & 0 & -1 & 2 \end{pmatrix}, \quad \mathbf{T}_2 = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & 0 & \dots & 0 \\ 0 & -1 & 2 & -1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & -1 & 2 & -1 & 0 \\ 0 & \dots & 0 & 0 & -1 & 2 & -1 \\ 0 & \dots & 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

represent two discrete lattice approximations to $-\partial_x^2$ on a finite interval.

a) What are the boundary conditions defining the domains of the corresponding continuum differential operators? [They are either Dirichlet ($y = 0$) or Neumann ($y' = 0$) boundary conditions.] Make sure you explain your reasoning.

b) Verify that

$$\begin{aligned} [\mathbf{T}_1^{-1}]_{ij} &= \min(i,j) - \frac{ij}{k+1}, \\ [\mathbf{T}_2^{-1}]_{ij} &= \min(i,j). \end{aligned}$$

c) Find the continuum Green functions for the boundary value problems approximated by the matrix operators. Compare each of the matrix inverses with its corresponding continuum Green function. Are they similar?

**4) Eigenfunction expansion** : We saw in class that the resolvent (Green function) $G_\lambda(x,x') = (L-\lambda)^{-1}_{xx'}$ can be expanded as

$$(L-\lambda)^{-1}_{xx'} = \sum_{\lambda_n} \frac{\varphi_n(x)\varphi_n(x')}{\lambda_n - \lambda},$$

where $\varphi_n(x)$ is the normalized eigenfunction corresponding to the eigenvalue $\lambda_n$. The resolvent therefore has a *pole* whenever $\lambda$ approaches $\lambda_n$. Consider the case

$$G_{\omega^2}(x,x') = \left(-\frac{d^2}{dx^2} - \omega^2\right)^{-1}_{xx'},$$

with boundary conditions $y(0) = y(L) = 0$. Using the explicit form of $G_{\omega^2}(x,x')$ that appears in Homework set 0

a) Confirm that $G_{\omega^2}$ becomes singular at exactly those values of $\omega^2$ corresponding to eigenvalues of $-\frac{d^2}{dx^2}$.

b) Confirm that the *residue* of the pole (the coefficient of $1/(\omega_n^2 - \omega^2)$) is precisely the product of the *normalized* eigenfunctions $\varphi_n(x)\varphi_n(x')$.