Phys 487 Discussion 15 – The 2 Standard Scattering Techniques of NRQM

Recap from last discussion: All of our scattering problems rest on the assumption that the scattering potential is central ($V(r)$) and falls off fast enough with r that the wavefunction in the asymptotic $r \to \infty$ region can be treated as that of a free particle. We thus write the total wavefunction of the incoming + outgoing particle as

$$\psi(r) \xrightarrow[r\to\infty]{} e^{ikz} + f(\theta)\frac{e^{ikr}}{r} \quad \text{with} \quad \hbar k = \sqrt{2\mu E} \; .$$

The given quantities are the reduced mass μ of the [beam + target] system and the kinetic energy E of the [beam + target] in the center-of-mass frame. Last week you analyzed the flow of probability represented by the above terms to derived the relation between the scattering amplitude $f(\theta)$ and the differential xsec :

$$\boxed{\frac{d\sigma}{d\Omega} = |\, f(\theta)\,|^2}$$

Today we derived the two standard ways of determining $f(\theta)$ in NRQM (Non-Relativistic Quantum Mech): the Born approximation and scattering phase shifts.

Method 1 : The Born Approximation

Using Fermi's Golden Rule from 1st-order perturbation theory, we derived this method of calculating the xsec:

$$\boxed{f(\theta) = \frac{\mu}{2\pi\hbar^2} V(q)}$$ where $\boxed{\vec{q} \equiv \vec{k}_i - \vec{k}_f}$ = momentum transfer from beam to target, and

$$\boxed{V(q) = \int d^3r \; V(r) e^{i\vec{q}\cdot\vec{r}}}$$ is the Fourier transform of the potential $V(r)$ created by the target.

This method is applicable when the potential $V(r)$ is << the system's kinetic energy.

Problem 1 : The Born Approximation takes on two delta functions *Qual Problem, of course* [1]

A free particle of mass m, travelling with momentum p parallel to the z-axis, scatters off the potential

$$V(r) = V_0\left[\delta(\vec{r} - \varepsilon\hat{z}) - \delta(\vec{r} + \varepsilon\hat{z})\right].$$

(a) Calculate the differential scattering cross section, $d\sigma/d\Omega$, in the Born approximation.

▶ GUIDANCE #1: Clearly you calculate the Fourier transform of the potential, $V(q) = \int d^3r \; V(r) e^{i\vec{q}\cdot\vec{r}}$, then get $f(\theta) = \frac{\mu}{2\pi\hbar^2} V(q)$, and finally get $\frac{d\sigma}{d\Omega} = |\, f(\theta)\,|^2$. The steps are obvious at least. :-)

One point that requires thought is that your $V(q)$ will end up with "$\vec{q}\cdot\hat{z}$" in a couple of places , and that needs to be expressed in terms of the <u>scattering angle θ</u>. The definition of the scattering angle is important here : it is the angle between the beam-particle direction $\vec{k}_i$ and the scattered-particle direction $\vec{k}_f$. The question tells us that the beam is aligned with the z axis, i.e. $\vec{k}_i = k_i\,\hat{z}$.

▶ GUIDANCE #2: The <u>magnitudes of $\vec{k}_i$ and $\vec{k}_f$</u> also need a little thought. This is **elastic scattering**: there is nowhere for any energy to go, and we are working in the CM frame (via the use of the reduced mass in our

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[1] **Q1** (a) $\frac{d\sigma}{d\Omega} = \frac{m^2 V_0^2}{\pi^2\hbar^4}\sin^2\left(2\varepsilon k \sin^2\frac{\theta}{2}\right)$ (b) skip

equations). If you think through that sentence, you will see that the magnitudes of the incoming and outgoing momenta must be the same : $\boldsymbol{k_i = k_f}$. This will be true in ≈ every NRQM scattering problem you encounter.

Method 2 : S-Wave Phase Shifts

• **Key Concept #1**: Here again is our master expression for an (incoming + scattered)-particle wavefunction in the free, asymptotic $r \to \infty$ region:

$$\psi(r) \xrightarrow[r\to\infty]{} e^{ikz} + f(\theta)\frac{e^{ikr}}{r}$$

As we learned in class, we can rewrite this expression as a **sum over partial waves**, which are solutions $R_l(r)Y_{l0}(\theta)$ of the free-particle Schrödinger Equ for a specific l value :

$$\psi(r) \xrightarrow[r\to\infty]{} \sum_l \psi_l(r,\theta) \;=\; \sum_l A_l P_l(\cos\theta)\left[e^{2i\delta_l}\frac{e_{\text{OUT}}^{i(kr-l\pi/2)}}{kr} - \frac{e_{\text{IN}}^{-i(kr-l\pi/2)}}{kr}\right] \quad \text{for some coefficients } A_l.$$

(Where did the Legendre polynomials come from? → We are only considering cases that are ϕ-independent, so the only spherical harmonics we keep are those with m=0, and the $Y_{l0}(\theta)$ are all proportional to $P_l(\cos\theta)$.)

The $e^{2i\delta_l}$ up there is the first heart of our strategy : each individual partial wave $\psi_l(r,\theta)$ is the sum of an incoming spherical wave and an outgoing spherical wave of the same amplitude that is merely phase-shifted by an amount δ_l to account for the effect of the scattering potential $V(r)$. Besides this phase-shift concept, you need one formula from the full phase-shift analysis: by matching the previous two expressions to each other (which is actually just algebra) you get the scattering amplitude $f(\theta)$ as a sum over Legendre polynomials, one for each partial wave :

$$\boxed{f(\theta) = \frac{1}{k}\sum_l (2l+1)e^{i\delta_l}\sin\delta_l\, P_l(\cos\theta)}$$

Since $d\sigma/d\Omega = |f(\theta)|^2$, find the phase shifts $\{\ \delta_l\ \}$ for the partial waves and you have found your xsec.

• **Key Concept #2**: When low-energy particles scatter from **short-range potentials** $V(r)$, only particles of low angular momentum will even *encounter* the potential and be scattered. Let the range of such a potential be a. To feel any effect at all from this potential, a beam particle must have impact parameter $b \le a$. The angular momentum of the beam relative to the origin is $L = |\vec{r}\times\vec{p}| = r_\perp p = bp$. To be scattered at all, the beam must have

$$L = \hbar\sqrt{l(l+1)} \le pa \quad \text{where } a \text{ is the range of the potential doing the scattering,}$$

$$\therefore\ \sqrt{l(l+1)} \le ka \quad \text{since } k=p/\hbar, \text{ as always.}$$

It is quite common with low-energy beams and short-range potentials that **only the s-wave** (l=0) **is scattered**. In such cases, we have only one partial wave to phase-shift. With only one, we can do everything by hand. With only the l=0 partial wave contributing to the scattering amplitude $f(\theta)$, the boxed formula above collapses to the only formula we actually need in this section:

$$\boxed{\frac{d\sigma}{d\Omega} = |f(\theta)|^2 = \left(\frac{\sin\delta_0}{k}\right)^2} \quad \text{for } s\text{-wave scattering.}$$

There you go: for low-energy scattering, find the s-wave phase shift δ_0 and you have your cross-section.

Problem 2 : S-Wave Phase Shifts take on an attractive shell *Qual Problems R Us* [2]

Consider a particle of mass m that scatters in three dimensions from a potential

$$V(r) = -C\,\delta\big(r - a\big)$$

where C and a are positive constants. (This represents an attractive contact potential that occupies a thin spherical shell of radius a. Like a round eggshell covered with superglue. Weird.)

Derive the exact s-wave expression for the total scattering cross section in the limit of very low particle energy.

This is the first time we are doing one of these, so let's write down some steps. <u>We have no convenient formula for figuring out phase shifts</u>. What we must do is <u>implement the phase shift idea by hand</u> for the one case <u>l=0</u> (the "s-wave").

(STEP a) Figure out the form of the radial wavefunction $R(r)$ for the case l=0 and the potential given in the question in the two regions relevant to this problem : $r > a$ and $r < a$. In other words, go back to 486 basics, literally: obtain the radial Schrödinger equation for this potential, set $l = 0$, and write down the general solutions for the regions $0 < r < a$ ("interior") and $r > a$ ("exterior"). Remember that, with radial equations, it is almost always way easier to work with <u>$u(r) \equiv r\,R(r)$</u> than with $R(r)$ itself. (By all means, go through notes or textbook to find the radial Schrödinger equation.)

(STEP b) We will have to apply **boundary conditions** to u_{INT} and u_{EXT}. Step-by-step … first, consider the point ***r*=0** $\rightarrow$ always an important point to consider in spherical coordinates! Remember that you need <u>$R(r) = u(r)\,/\,r$</u> = the *real* wavefunction to be <u>well-behaved at the origin</u>. Does that limit your choices from (a)?

(STEP c) Now we apply the **phase shift idea**: <u>The entire effect of the potential</u> = the sticky-eggshell δ-function potential at $r = a$ is to simply <u>insert a phase shift of δ</u> <u>between the free wavefunctions on either side</u> of the non-free region $r = a$. Adjust your u_{INT} and u_{EXT} solutions if necessary to accomplish that. Remember that this phase-shift concept is that for each partial wave independently — i.e. for each l independently (here we only have one, l=0) — the flux of probability into a region with non-zero potential is equal to the flux of probability out of the region of non-zero potential, because there is nothing in a central-potential problem that can move probability from one l-state to a different l-state.

(STEP d) Now apply **standard boundary conditions at *r*=*a***. Totally a 486 thing!
Hint 1: This requires a glance at the 486 formula sheet because of the δ function at r=a.
Hint 2: You will find it useful to introduce the symbol $\gamma \equiv 2mC\,/\,\hbar^2$.
Hint 3: You will get a transcendental equation that you can't solve just yet, and that's ok.

(STEP e) We hit a transcendental equation, what futher approximations can we make? … AH, the question said to derive the s-wave total cross section **in the limit of very low particle energy**. Low energy means low momentum, so take the approximation $k \approx 0$ wherever you can (without losing k entirely of course!) Apply that limit to get an expression for the phase shift δ in terms of the given quantities k, γ, and a.

(STEP f) We have our phase shift, what do we do with it? We get the cross-section! Apply the last boxed formula above to get $d\sigma/d\Omega$, then integrate it over angles to get the total cross-section σ.

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[2] **Q2 (a)** $u_{\text{INT}}(r) = A\sin(kr + \alpha)$; $u_{\text{EXT}}(r) = B\sin(kr + \beta)$ where A, B, α, β are the free parameters that are always present in the general solutions of 2nd-order ODEs! **(b)** $R_{\text{INT}}(r) = u_{\text{INT}}(r)/r = A\sin(kr+\alpha)/r \rightarrow$ non-infinite as $r \rightarrow 0$ as long as α=0. $\therefore\ u_{\text{INT}}(r) = A\sin(kr)$. **(c)** $u_{\text{INT}}(r) = A\sin(kr)$; $u_{\text{EXT}}(r) = B\sin(kr + \delta)$ **(d)** boundary conditions give $\tan(ka+\delta) = \tan(ka)\,/\,[\,1 - (\gamma/k)\tan(ka)\,]$ **(e)** $\delta \approx k\,\gamma\,a^2\,/\,(1-\gamma a) \ll 1$ for very small k **(f)** $d\sigma/d\Omega = \sin^2\delta\,/\,k^2 \approx \delta^2\,/\,k^2$ = constant so $\sigma = 4\pi\, d\sigma/d\Omega \approx 4\pi\,\gamma^2 a^4\,/\,(1-\gamma a)^2$