

## Phys 487 Discussion 13 – Spontaneous Emission

Here is Fermi's Golden Rule :  $R_{i \rightarrow f} = \frac{2\pi}{\hbar} |V_{fi}|^2 n(E_f)$  transitions per unit time from state  $i$  to state  $f$ .

We applied this to electric dipole **E1 radiation** : an oscillating potential energy  $H' = V(r) [e^{-i\omega t} + e^{+i\omega t}]$  caused by an approximately-uniform electric field  $\vec{E}_0$ , which gives  $V(\vec{r}) = -q\vec{E}_0 \cdot \vec{r}$ . We calculated the ingredients of Fermi's golden rule for E1 radiation incident on an electron and found :

- $V_{fi} \neq 0$  only when the **selection rules** at right are satisfied

- $|V_{fi}|^2 = \frac{E_0^2 e^2}{3} |\langle f | \vec{r} | i \rangle|^2 = \frac{E_0^2 e^2}{3} |\vec{r}_{fi}|^2$  for incoherent radiation, i.e. random direction & polarization

- $n(E_f) = \frac{L^3 \omega^2}{\pi^2 \hbar c^3} = \#$  of available (possible) photon states of frequency  $\omega$  per unit energy

*For the electron making the transition*

- (a)  $\Delta l = \pm 1$
- (b)  $\Delta m_l = 0, \pm 1$

*For the atom as a whole*

- (a)  $\Delta S = 0$
- (b)  $\Delta L = 0, \pm 1$  ( $L = 0 \leftrightarrow L' = 0$  forbidden)
- (c)  $\Delta M_L = 0, \pm 1$
- (d)  $\Delta J = 0, \pm 1$  ( $J = 0 \leftrightarrow J' = 0$  forbidden)
- (e)  $\Delta M_J = 0, \pm 1$

(We were TWO MINUTES away from deriving that last one in class, you'll get an email when the conclusion recording is posted!) Let's plug these two ingredients into F.G.R. and set  $|\omega| = |\omega_{fi}| \equiv |E_i - E_f| / \hbar$  to satisfy energy conservation for the absorption/emission of a photon of energy  $\hbar\omega$ ,  $E_i - E_f = \pm \hbar\omega$ . We then obtain the following transition rate, which is so hugely important it gets a special name :

$$A = \frac{\omega_{fi}^3 e^2 |\vec{r}_{fi}|^2}{3\pi\epsilon_0 \hbar c^3} = \text{Einstein's A coefficient where } \omega_{fi} \equiv (E_i - E_f) / \hbar$$

This "A" is the number of transitions / second of **spontaneous emission** of E1 radiation from an atom in state  $i$  going to state  $f$ . IT REALLY SHOULD BE LABELLED  $A_{i \rightarrow f}$ . We'll call it  $A_{i \rightarrow f}$  whenever it improves clarity.

Since "A" is the rate of *spontaneous* transition from a state  $i$  to a state  $f$ , it is precisely related to the **natural lifetime**  $\tau$  of state  $i$ , i.e. how long it will remain in state  $i$  before dropping down to state  $f$ . The relationship is :

$$\tau = \frac{1}{A} = \text{lifetime of state } i = \text{time for the \# of atoms in state } i \text{ to drop by factor } 1/e$$

To be exact, this lovely relation is for the lifetime of state  $i$  under the transition  $i \rightarrow f$  for a specific final state  $f$  (since "A" refers to both a specific initial state and a specific final state). Usually, for a given initial state  $i$  there is a very dominant transition  $i \rightarrow f$  that is much more probably (higher rate) than all the others, but if there are several final states to which state  $i$  can decay, the lifetimes add as follows :

$$\tau_i = \frac{1}{\sum_f A_{i \rightarrow f}} \text{ i.e. the rates } A_{i \rightarrow f} \text{ add, not the lifetimes, which makes total sense. } \odot$$

All details of the above (why is A the *spontaneous* transition rate? why is  $1/A$  the time it takes for a sample to drop to  $1/e$ ? how is this derivation related to the stat-mech/thermo one in Griffiths? other questions you text me?) will be discussed in the upcoming E1-radiation-wrapup recording.

### Problem 1 : Reduced Mass

Griffiths 5.2

In classical mechanics, two bodies with masses  $m_1$  &  $m_2$  with a central force between them can be treated as ONE EFFECTIVE body with the **reduced mass**  $\mu = m_1 m_2 / (m_1 + m_2)$ . (Griffiths problem 5.1 goes through the demonstration that this simplification carries over directly to quantum mechanics, if you would like to work through it; you will see that the Schrödinger equation simplifies exactly as Lagrange's equations did in CM, not too surprising.) We can use this concept to correct for the motion of the nucleus in hydrogen : we simply replace the electron mass with the reduced mass for the (electron + proton) system.

- Find (to two significant digits) the percent error in the binding energy of hydrogen introduced by our use of  $m$  instead of  $\mu$ .
- Find the separation in wavelength between the red "Balmer" lines ( $n=3 \rightarrow n=2$ ) for hydrogen & deuterium.
- Suppose you wanted to confirm the existence of **muonic hydrogen**, in which the electron is replaced by a muon (same charge, same spin, but 207 times heavier). At what wavelength would you look for the "Lyman- $\alpha$ " line ( $n=2 \rightarrow n=1$ )?

### Problem 2 : Positronium

part of a Qual Problem

Positronium is a hydrogen-like bound state made up of an electron and a positron.

- Estimate the binding energy of the ground state ( $n = 1$ ) and the Lyman- $\alpha$  ( $2p \rightarrow 1s$ ) transition wavelength for positronium. HINT: Problem 1 is highly useful here!
- The lifetime for the decay from  $2p$  to  $1s$  for the hydrogen atom is 1.6 ns. Estimate the lifetime for the same decay in positronium.

### Problem 3 : Various Elements

Qual Problem

- List the ground-state electronic configurations and the  $L$ ,  $S$ , and  $J$  quantum numbers for the following atoms: Li ( $Z=3$ ), B ( $Z=5$ ), N ( $Z=7$ ), Na ( $Z=11$ ), K ( $Z=19$ ). HINT: It is implied that you should specify  $^{2S+1}L_J$  for the *ground states* of these atoms. (There is no other way to interpret the question that would give a unique answer for each atom!)
- The lowest frequency line in the absorption spectrum of Na is a doublet. What mechanism splits the corresponding pair of energy levels? The splitting between levels is proportional to  $\langle r^n \rangle$  where  $r$  is the distance of the valence electron from the nucleus. What is the numerical value of  $n$ ? HINT: You can only figure out  $n$  once you know the mechanism. You *may* have to hunt through notes / Griffiths / web to find the formula you need for  $n$ , it is not on our formula sheet directly, but I bet you can figure it out on your own!
- Consider potassium ( $Z=19$ ). Make a level diagram indicating the three lowest-lying states  $^{2S+1}L_J$  in energy order. (Check your result against the NIST database!) Then consider the effect of adding a weak magnetic field  $B$ ; this will further split the levels  $\rightarrow$  how? (HINT: you will get a new level diagram with  $M_J$  now specified as well; it should have 8 levels in total).
- On your 8-level diagram, indicate all the allowed transitions.  
HINT: you should find 10 allowed transitions, where as you know, "allowed" is code for "E1 transitions".)