1D SHO $\hat{H}(x)=\frac{1}{2m}\left(\hat{p}^2+m^2\omega^2x^2\right)$ Define $x_0\equiv\sqrt{\frac{\hbar}{m\omega}}$, $\xi\equiv\frac{x}{x_0}$ $\rightarrow$ $\hat{H}(\xi)=\frac{\hbar\omega}{2}\left(\xi^2-\frac{d^2}{d\xi^2}\right)$

$$E_n=\left(n+\tfrac{1}{2}\right)\hbar\omega\,,\quad \psi_n(x)=\left(\frac{1}{\pi x_0^2}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}\,H_n\!\left(\frac{x}{x_0}\right)e^{-\frac{x^2}{2x_0^2}}$$

where $H_0(\xi)=1$, $H_2(\xi)=4\xi^2-2$, $H_1(\xi)=2\xi$, $H_3(\xi)=8\xi^3-12\xi$, $H_n(\xi)=(-1)^n e^{\xi^2}\frac{d^n}{d\xi^n}e^{-\xi^2}$

$$\hat{a}_\pm=\frac{1}{\sqrt{2}}\left(\xi\mp\frac{d}{d\xi}\right)\rightarrow\begin{array}{l}\hat{a}_+\psi_n=\sqrt{n+1}\,\psi_{n+1}\\ \hat{a}_-\psi_n=\sqrt{n}\,\psi_{n-1}\end{array},\quad \begin{array}{l}\hat{H}=\hbar\omega\left(\hat{a}_+\hat{a}_-+\tfrac{1}{2}\right)\\ \left[\hat{a}_-,\hat{a}_+\right]=1\end{array}$$

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for $-8/15$ read $-\sqrt{8/15}$.

Notation:

		J	J	...
		M	M	...
m_1	m_2	Coefficients		
m_1	m_2			
.	.			

$$Y_1^0=\sqrt{\frac{3}{4\pi}}\cos\theta$$
$$Y_1^1=-\sqrt{\frac{3}{8\pi}}\sin\theta\,e^{i\phi}$$
$$Y_2^0=\sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta-\frac{1}{2}\right)$$
$$Y_2^1=-\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\,e^{i\phi}$$
$$Y_2^2=\frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\,e^{2i\phi}$$
$$Y_\ell^{-m}=(-1)^m Y_\ell^{m*}$$

$$\langle j_1j_2m_1m_2|j_1j_2JM\rangle=(-1)^{J-j_1-j_2}\langle j_2j_1m_2m_1|j_2j_1JM\rangle$$

1/2×1/2

m_1	m_2	J=1, M=+1
+1/2	+1/2	1

m_1	m_2	J=1, M=0	J=0, M=0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	J=1, M=−1
−1/2	−1/2	1

1×1/2

m_1	m_2	J=3/2, M=+3/2
+1	+1/2	1

m_1	m_2	J=3/2, M=+1/2	J=1/2, M=+1/2
+1	−1/2	1/3	2/3
0	+1/2	2/3	−1/3

m_1	m_2	J=3/2, M=−1/2	J=1/2, M=−1/2
0	−1/2	2/3	1/3
−1	+1/2	1/3	−2/3

m_1	m_2	J=3/2, M=−3/2
−1	−1/2	1

2×1/2

m_1	m_2	J=5/2, M=+5/2
+2	+1/2	1

m_1	m_2	J=5/2, M=+3/2	J=3/2, M=+3/2
+2	−1/2	1/5	4/5
+1	+1/2	4/5	−1/5

m_1	m_2	J=5/2, M=+1/2	J=3/2, M=+1/2
+1	−1/2	2/5	3/5
0	+1/2	3/5	−2/5

m_1	m_2	J=5/2, M=−1/2	J=3/2, M=−1/2
0	−1/2	3/5	2/5
−1	+1/2	2/5	−3/5

m_1	m_2	J=5/2, M=−3/2	J=3/2, M=−3/2
−1	−1/2	4/5	1/5
−2	+1/2	1/5	−4/5

m_1	m_2	J=5/2, M=−5/2
−2	−1/2	1

3/2×1/2

m_1	m_2	J=2, M=+2
+3/2	+1/2	1

m_1	m_2	J=2, M=+1	J=1, M=+1
+3/2	−1/2	1/4	3/4
+1/2	+1/2	3/4	−1/4

m_1	m_2	J=2, M=0	J=1, M=0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	J=2, M=−1	J=1, M=−1
−1/2	−1/2	3/4	1/4
−3/2	+1/2	1/4	−3/4

m_1	m_2	J=2, M=−2
−3/2	−1/2	1

2×1

m_1	m_2	J=3, M=+3
+2	+1	1

m_1	m_2	J=3, M=+2	J=2, M=+2
+2	0	1/3	2/3
+1	+1	2/3	−1/3

m_1	m_2	J=3, M=+1	J=2, M=+1	J=1, M=+1
+2	−1	1/15	1/3	3/5
+1	0	8/15	1/6	−3/10
0	+1	2/5	−1/2	1/10

m_1	m_2	J=3, M=0	J=2, M=0	J=1, M=0
+1	−1	1/5	1/2	3/10
0	0	3/5	0	−2/5
−1	+1	1/5	−1/2	3/10

m_1	m_2	J=3, M=−1	J=2, M=−1	J=1, M=−1
0	−1	2/5	1/2	1/10
−1	0	8/15	−1/6	−3/10
−2	+1	1/15	−1/3	3/5

m_1	m_2	J=3, M=−2	J=2, M=−2
−1	−1	2/3	1/3
−2	0	1/3	−2/3

m_1	m_2	J=3, M=−3
−2	−1	1

3/2×1

m_1	m_2	J=5/2, M=+5/2
+3/2	+1	1

m_1	m_2	J=5/2, M=+3/2	J=3/2, M=+3/2
+3/2	0	2/5	3/5
+1/2	+1	3/5	−2/5

m_1	m_2	J=5/2, M=+1/2	J=3/2, M=+1/2	J=1/2, M=+1/2
+3/2	−1	1/10	2/5	1/2
+1/2	0	3/5	1/15	−1/3
−1/2	+1	3/10	−8/15	1/6

m_1	m_2	J=5/2, M=−1/2	J=3/2, M=−1/2	J=1/2, M=−1/2
+1/2	−1	3/10	8/15	1/6
−1/2	0	3/5	−1/15	−1/3
−3/2	+1	1/10	−2/5	1/2

m_1	m_2	J=5/2, M=−3/2	J=3/2, M=−3/2
−1/2	−1	3/5	2/5
−3/2	0	2/5	−3/5

m_1	m_2	J=5/2, M=−5/2
−3/2	−1	1

1×1

m_1	m_2	J=2, M=+2
+1	+1	1

m_1	m_2	J=2, M=+1	J=1, M=+1
+1	0	1/2	1/2
0	+1	1/2	−1/2

m_1	m_2	J=2, M=0	J=1, M=0	J=0, M=0
+1	−1	1/6	1/2	1/3
0	0	2/3	0	−1/3
−1	+1	1/6	−1/2	1/3

m_1	m_2	J=2, M=−1	J=1, M=−1
0	−1	1/2	1/2
−1	0	1/2	−1/2

m_1	m_2	J=2, M=−2
−1	−1	1

3/2×3/2

m_1	m_2	J=3, M=+3
+3/2	+3/2	1

m_1	m_2	J=3, M=+2	J=2, M=+2
+3/2	+1/2	1/2	1/2
+1/2	+3/2	1/2	−1/2

m_1	m_2	J=3, M=+1	J=2, M=+1	J=1, M=+1
+3/2	−1/2	1/5	1/2	3/10
+1/2	+1/2	3/5	0	−2/5
−1/2	+3/2	1/5	−1/2	3/10

m_1	m_2	J=3, M=0	J=2, M=0	J=1, M=0	J=0, M=0
+3/2	−3/2	1/20	1/4	9/20	1/4
+1/2	−1/2	9/20	1/4	−1/20	−1/4
−1/2	+1/2	9/20	−1/4	−1/20	1/4
−3/2	+3/2	1/20	−1/4	9/20	−1/4

m_1	m_2	J=3, M=−1	J=2, M=−1	J=1, M=−1
+1/2	−3/2	1/5	1/2	3/10
−1/2	−1/2	3/5	0	−2/5
−3/2	+1/2	1/5	−1/2	3/10

m_1	m_2	J=3, M=−2	J=2, M=−2
−1/2	−3/2	1/2	1/2
−3/2	−1/2	1/2	−1/2

m_1	m_2	J=3, M=−3
−3/2	−3/2	1

2×3/2

m_1	m_2	J=7/2, M=+7/2
+2	+3/2	1

m_1	m_2	J=7/2, M=+5/2	J=5/2, M=+5/2
+2	+1/2	3/7	4/7
+1	+3/2	4/7	−3/7

m_1	m_2	J=7/2, M=+3/2	J=5/2, M=+3/2	J=3/2, M=+3/2
+2	−1/2	1/7	16/35	2/5
+1	+1/2	4/7	1/35	−2/5
0	+3/2	2/7	−18/35	1/5

m_1	m_2	J=7/2, M=+1/2	J=5/2, M=+1/2	J=3/2, M=+1/2	J=1/2, M=+1/2
+2	−3/2	1/35	6/35	2/5	2/5
+1	−1/2	12/35	5/14	0	−3/10
0	+1/2	18/35	−3/35	−1/5	1/5
−1	+3/2	4/35	−27/70	2/5	−1/10

m_1	m_2	J=7/2, M=−1/2	J=5/2, M=−1/2	J=3/2, M=−1/2	J=1/2, M=−1/2
+1	−3/2	4/35	27/70	2/5	1/10
0	−1/2	18/35	3/35	−1/5	−1/5
−1	+1/2	12/35	−5/14	0	3/10
−2	+3/2	1/35	−6/35	2/5	−2/5

m_1	m_2	J=7/2, M=−3/2	J=5/2, M=−3/2	J=3/2, M=−3/2
0	−3/2	2/7	18/35	1/5
−1	−1/2	4/7	−1/35	−2/5
−2	+1/2	1/7	−16/35	2/5

m_1	m_2	J=7/2, M=−5/2	J=5/2, M=−5/2
−1	−3/2	4/7	3/7
−2	−1/2	3/7	−4/7

m_1	m_2	J=7/2, M=−7/2
−2	−3/2	1

2×2

m_1	m_2	J=4, M=+4
+2	+2	1

m_1	m_2	J=4, M=+3	J=3, M=+3
+2	+1	1/2	1/2
+1	+2	1/2	−1/2

m_1	m_2	J=4, M=+2	J=3, M=+2	J=2, M=+2
+2	0	3/14	1/2	2/7
+1	+1	4/7	0	−3/7
0	+2	3/14	−1/2	2/7

m_1	m_2	J=4, M=+1	J=3, M=+1	J=2, M=+1	J=1, M=+1
+2	−1	1/14	3/10	3/7	1/5
+1	0	3/7	1/5	−1/14	−3/10
0	+1	3/7	−1/5	−1/14	3/10
−1	+2	1/14	−3/10	3/7	−1/5

m_1	m_2	J=4, M=0	J=3, M=0	J=2, M=0	J=1, M=0	J=0, M=0
+2	−2	1/70	1/10	2/7	2/5	1/5
+1	−1	8/35	2/5	1/14	−1/10	−1/5
0	0	18/35	0	−2/7	0	1/5
−1	+1	8/35	−2/5	1/14	1/10	−1/5
−2	+2	1/70	−1/10	2/7	−2/5	1/5

m_1	m_2	J=4, M=−1	J=3, M=−1	J=2, M=−1	J=1, M=−1
+1	−2	1/14	3/10	3/7	1/5
0	−1	3/7	1/5	−1/14	−3/10
−1	0	3/7	−1/5	−1/14	3/10
−2	+1	1/14	−3/10	3/7	−1/5

m_1	m_2	J=4, M=−2	J=3, M=−2	J=2, M=−2
0	−2	3/14	1/2	2/7
−1	−1	4/7	0	−3/7
−2	0	3/14	−1/2	2/7

m_1	m_2	J=4, M=−3	J=3, M=−3
−1	−2	1/2	1/2
−2	−1	1/2	−1/2

m_1	m_2	J=4, M=−4
−2	−2	1

Atomic Structure

Bohr magneton $\mu_B=\frac{e\hbar}{2m_e}$

g factor $\vec{\mu}_L=\frac{e\hbar}{m}\vec{L}$, $\vec{\mu}_S=g\frac{e\hbar}{m}\vec{S}$, $g_{\text{spin-1/2 point particle}}=2$

Hund rules 1. Max S 2. Max L
3. Min J for $\le 1/2$-filled shells

Perturbation Theory – Time-Independent

$H = H_0 + H'$

- H_0 solvable w eigen-* $\{E_n^{(0)}\},\{|n^{(0)}\rangle\}$
- $H' \ll H_0$

Expansions for eigen-* of H: $E_n = E_n^{(0)} + E_n^{(1)} + \cdots$ & $|n\rangle = |n^{(0)}\rangle + |n^{(1)}\rangle + \cdots$

For a **non-degenerate** eigenvalue $E_n^{(0)}$ of H_0: $|n^{(1)}\rangle = \sum_{m \neq n} \frac{H'_{mn}}{E_n^{(0)} - E_m^{(0)}} |m^{(0)}\rangle$ with $H'_{mn} \equiv \langle m^{(0)}|H'|n^{(0)}\rangle$

$$E_n^{(j)} = \langle n^{(0)}|H'|n^{(j-1)}\rangle \quad \to \quad E_n^{(1)} = H'_{nn}\,, \quad E_n^{(2)} = \sum_{m \neq n} \frac{|H'_{mn}|^2}{E_n^{(0)} - E_m^{(0)}}$$

For a **degenerate** eigenvalue $E_D^{(0)}$ of H_0:

- Let $\{ |\alpha_1^{(0)}\rangle, ..., |\alpha_n^{(0)}\rangle \}$ = degenerate subspace D sharing eigenvalue $E_D^{(0)}$
- Find $\{ |\beta_1^{(0)}\rangle, ..., |\beta_n^{(0)}\rangle \}$ = eigenvectors of H' within subspace D

  = linear combinations of $|\alpha_i^{(0)}\rangle$ states that diagonalize $\mathbf{H'}$

$\Rightarrow$ 1st order energy correction is $E_{\beta i}^{(1)} = \langle \beta_i^{(0)}|H'|\beta_i^{(0)}\rangle$

↓ *not on Midterm 1* ↓

Variational Principle

$E_{gs} \leq \langle \psi|H|\psi\rangle \;\; \forall\, \psi$

Sudden / Adiabatic Approx

ψ / n unchanged by ΔH

WKB Approximation

In **A**llowed ($V<E$) & **B**locked ($V>E$) regions, with $p(x) \equiv \sqrt{2m(E - V(x))}$,

Solution forms: $\psi_{\mathbf{A}}(x) = \frac{A}{\sqrt{p(x)}} \exp\left[\pm i \int^x \frac{p(x')}{\hbar} dx'\right]$, $\quad \psi_{\mathbf{B}}(x) = \frac{B}{\sqrt{|p(x)|}} \exp\left[\pm \int^x \frac{|p(x')|}{\hbar} dx'\right]$

Connection formulae at turning points $x = a$: with $\int k \equiv \int p(x')/\hbar \; dx'$ & "barrier" $\equiv V > E$ region

barrier on LEFT ($x<a$):

$$\psi_{\mathbf{B}}(x) = \frac{1}{2}\frac{C}{\sqrt{|p|}} \exp\left[-\int_x^a |k|\right] \text{ matches to } \psi_{\mathbf{A}}(x) = \frac{C}{\sqrt{p}} \cos\left[\int_a^x k - \frac{\pi}{4}\right]$$

$$\psi_{\mathbf{B}}(x) = \frac{1}{2}\frac{D}{\sqrt{|p|}} \exp\left[+\int_x^a |k|\right] \text{ matches to } \psi_{\mathbf{A}}(x) = \frac{D}{\sqrt{p}} \cos\left[\int_a^x k + \frac{\pi}{4}\right]$$

barrier on RIGHT ($x>a$):

$$\psi_{\mathbf{B}}(x) = \frac{1}{2}\frac{C}{\sqrt{|p|}} \exp\left[-\int_a^x |k|\right] \text{ matches to } \psi_{\mathbf{A}}(x) = \frac{C}{\sqrt{p}} \cos\left[\int_x^a k - \frac{\pi}{4}\right]$$

$$\psi_{\mathbf{B}}(x) = \frac{1}{2}\frac{D}{\sqrt{|p|}} \exp\left[+\int_a^x |k|\right] \text{ matches to } \psi_{\mathbf{A}}(x) = \frac{D}{\sqrt{p}} \cos\left[\int_x^a k + \frac{\pi}{4}\right]$$

"Barrier on **right**" formulae are **IDENTICAL** to "barrier on **left**" ones except that the **order of the integral bounds is reversed**. In ALL cases, the lower bound is at smaller x than the upper bound.

Perturbation Theory – Time Dependent

- $H(t) = H^{(0)} + H'(t)$
- $\{ E_n^{(0)}, |n^{(0)}\rangle \}$ = the eigen-* of $H^{(0)}$
- $\omega_{fn} \equiv \left(E_f^{(0)} - E_n^{(0)}\right)/\hbar$
- $H'_{fn} \equiv \langle f^{(0)}|H'|n^{(0)}\rangle$

$|\psi(t)\rangle = \sum_n c_n(t)\, e^{-i\omega_n t} |n^{(0)}\rangle$ where $i\hbar \dot{c}_f(t) = \sum_n H'_{fn}\, e^{i\omega_{fn} t} c_n(t)$

To 1st order in $H' \ll H^{(0)}$, with $|\psi(t_0)\rangle = |i^{(0)}\rangle$: $c_f(t) \approx \delta_{fi} + \frac{1}{i\hbar}\int_{t_0}^t H'_{fi}(t') e^{i\omega_{fi} t'} dt' \quad \to \quad P_{i\to f} = |c_f(t)|^2$

Fermi's Golden Rule: $R_{i\to f} \equiv \frac{P_{i\to f}}{t} = \frac{2\pi}{\hbar} |V_{fi}|^2 \, n(E_f)$