# NPRE 441 Spring 2023
# Quiz 2 Solutions

(02-27-23)

**Question 1: Compton scattering of photons (25 points)**

The Klein-Nishina equation below describes the differential cross section of an electron for Compton scattering.

$$\frac{d\sigma}{d\Omega}(\theta) = r_e^2\left(\frac{1}{1+\alpha(1-\cos\theta)}\right)^2\left(\frac{1+\cos^2\theta}{2}\right)\left(1+\frac{\alpha^2(1-\cos\theta)^2}{(1+\cos^2\theta)[1+\alpha(1-\cos\theta)]}\right)\ (m^2sr^{-1})$$

$$,where\ \alpha = \frac{hv}{m_0c^2}\ \text{ and }\ r_e = \frac{k_0e^2}{m_0c^2}\ \text{is the classic electron radius } (2.818\ \times\ 10^{-15}\text{ m})$$

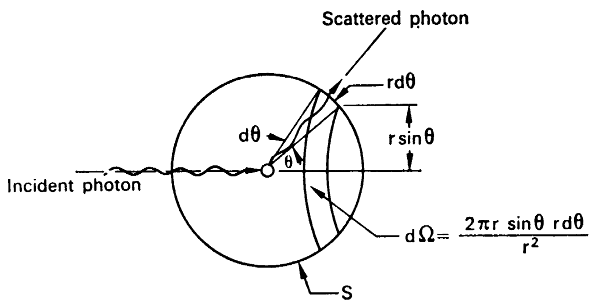
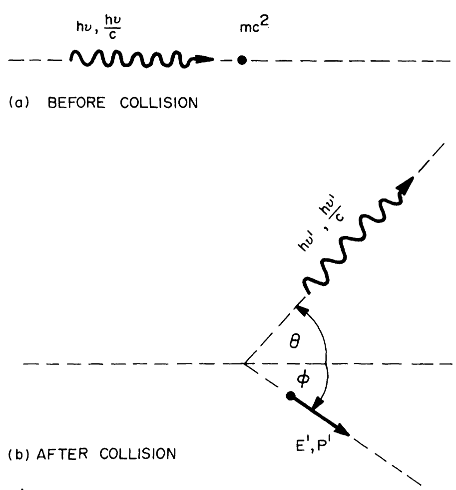



FIG. 5.15. Compton scattering diagram to illustrate differential scattering cross section. *S* is a sphere of unit radius whose center is the scattering electron.

For a gamma ray that undergoes a Compton scattering, the energy carried by the scattered gamma ray is given by the equation below.

$$hv' = \frac{hv}{1+\frac{hv}{mc^2}(1-\cos\theta)}$$




Please write down the equations for evaluating **the average energy transfer from the incident gamma-ray to the recoil electron through a single Compton scattering**.

**Solutions:**

**Part A: Derivation of the Compton equation**

From the conservation of total energy,

$$hv + m_e c^2 = hv' + E' \qquad (1)$$

From conservation of momentum in the x-direction,

$$\frac{hv}{c} = \frac{hv'}{c}cos\theta + P'cos\emptyset \qquad (2)$$

From conservation of momentum in the y-direction,

$$\frac{hv\prime}{c}sin\theta = P'sin\emptyset \qquad (3)$$

Rearrange Equation 2, then square Equation 2 and Equation 3, then add them together

$$\left(\frac{hv}{c} - \frac{hv'}{c}cos\theta\right)^2 = (P'cos\emptyset)^2, \qquad \left(\frac{hv'}{c}sin\theta\right)^2 = (P'sin\emptyset)^2$$

$$P'^2 = \left(\frac{hv}{c}\right)^2 + \left(\frac{hv'}{c}\right)^2 - \frac{2h^2vv'}{c^2}cos\theta \qquad (4)$$

Converting the momentum back to kinetic energy,

$$E' = \sqrt{m^2c^4 + P'^2c^2} \qquad (5)$$

Rearrange and square Equation 1,

$$E' = hv + mc^2 - hv', \text{ and}$$

$$E'^2 = m^2c^4 + h^2(v - v')^2 + 2mc^2h(v - v') \qquad (6)$$

Square Equation 5 and equate that to Equation 6,

$$E'^2 = m^2c^4 + P'^2c^2 = m^2c^4 + h^2(v - v')^2 + 2mc^2h(v - v') \qquad (7)$$

Plug in Equation 4 into Equation 7, then rearrange,

$$m^2c^4 + (hv)^2 + (hv')^2 - 2h^2vv'cos\theta = m^2c^4 + h^2(v - v')^2 + 2mc^2h(v - v'),$$

$$-2h^2vv'cos\theta = -2h^2vv' + 2mc^2h(v - v'),$$

$$\frac{h^2vv'}{mc^2}(1 - cos\theta) = h(v - v'),$$

$$hv' + hv'\left(\frac{hv}{mc^2}(1 - cos\theta)\right) = hv$$

$$hv' = \frac{hv}{1+\frac{hv}{mc^2}(1-cos\theta)} \qquad (8)$$

**Part B: The equations for evaluating the average energy transfer from the incident gamma-ray to the recoil electron through a single Compton scattering.**

The differential cross section for Compton scatter in respect to the energy transferred to the recoiled electron can be written as

$$\frac{d\sigma}{dE_{recoil}} = \frac{d\sigma}{d\Omega} * \frac{d\Omega}{d\theta} * \frac{d\theta}{dE_{recoil}} \quad , \tag{1}$$

where $\frac{d\sigma}{d\Omega}$ is given by the Klein-Nishina formula

$$\frac{d\sigma}{d\Omega}(\theta) = r_e^2 \left( \frac{1}{1+\alpha(1-\cos\theta)} \right)^2 \left( \frac{1+\cos^2\theta}{2} \right) \left( 1 + \frac{\alpha^2 (1-\cos\theta)^2}{(1+\cos^2\theta)[1+\alpha(1-\cos\theta)]} \right) \left( m^2 sr^{-1} \right)$$

Since, $d\Omega = \frac{2\pi r \sin\theta r d\theta}{r^2}$,

then we have $\frac{d\Omega}{d\theta} = 2\pi sin\theta.$ (2)

From Compton scattering equation solved in problem 1,

$$E_{recoil} = hv - hv' = hv - \frac{hv}{1+\frac{hv}{m_e c^2}(1-cos\theta)},$$

$$\frac{dE_{recoil}}{d\theta} = hv\left[1 + \frac{hv}{m_e c^2}(1 - cos\theta)\right]^{-2} \left(\frac{hv}{m_e c^2} sin\theta\right),$$

$$\frac{d\theta}{dE_{recoil}} = \frac{m_e c^2}{(hv)^2 sin\theta}\left[1 + \frac{hv}{m_e c^2}(1 - cos\theta)\right]^2 = -\frac{m_e c^2}{(hv - E_{recoil})^2 sin\theta} \tag{3}$$

Then, $\frac{d\sigma}{dE_{recoil}}$ could be written as explicit function of $E_{recoil}$.

$$\frac{d\theta}{dE_{recoil}}(\theta) \to \frac{d\sigma}{dE_{recoil}}(E_{recoil}) \tag{4}$$

Average recoil electron energy $E_{avg\_recoil}$ is of special interest since it is the amount of energy that the gamma-ray gives to a recoil electron through Compton scattering.

$$\frac{E_{avg_{recoil}}}{hv} = \int_{E_{recoil}} \frac{E_{recoil}}{hv} * \left[\frac{\left(\frac{d\sigma}{dE_{recoil}}\right)}{\sigma}\right] dE_{recoil}$$

Thus, the average recoil electron energy, or average gamma-ray energy loss, is:

$$E_{avg_{recoil}} = hv \int_{E_{recoil}} \frac{E_{recoil}}{hv} * \left[\frac{\left(\frac{d\sigma}{dE_{recoil}}\right)}{\sigma}\right] dE_{recoil}$$

**Question 2: A few conceptual questions (25 points)**

**(A)** What is the so-called **restricted stopping power** for heavy charged particles in a given absorbing media?

**(B)** What is the **energy transfer coefficient**? and what is the **energy absorption coefficient**? Please write down the equations for these attenuation coefficients for X-rays and gamma-rays and explain the meaning of each individual term in the equations.

**Solutions:**

**Part (A)**:

Restrict stopping power of heavy charged particles is the linear stopping power resultant from all the collisions between the heavy charged particle with electrons that led to energy transfer smaller than a given threshold.

**Part (B):**

Energy Transfer Coefficient: the fraction of energy that was originally carried by the incident gamma-ray beam and transferred into the kinetic energy of secondary electrons per unit distance of travel.

The fraction of energy that is carried away by characteristic x-rays following the photoelectric effect.

The fraction of energy that is carried away by the two 511 keV gamma rays generated by the annihilation of the positron

$$\frac{\mu_{tr}}{\rho} = \frac{\tau}{\rho}\left(1 - \frac{\delta}{hv}\right) + \frac{\sigma}{\rho}\left(\frac{E_{avg}}{hv}\right) + \frac{k}{\rho}\left(1 - \frac{2m_e c^2}{hv}\right)$$

The fraction of energy that is transferred to recoil electron through Compton scattering

Energy Absorption Coefficient: the fraction of energy that was originally carried by the incident gamma-ray beam and eventually absorbed inside the absorber.

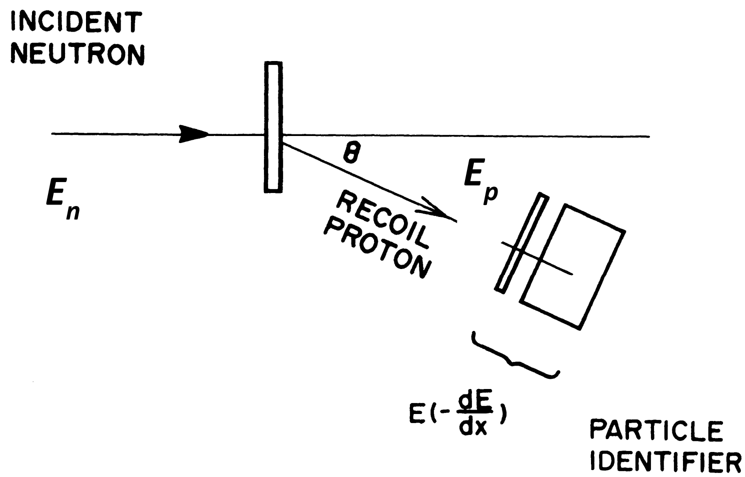
$$\frac{\mu_{en}}{\rho} = \frac{\mu_{tr}}{\rho}(1 - g)$$

Total mass energy transfer coefficient

Average fraction of energy of the initial kinetic energy transferred to electrons that is subsequently emitted as bremsstrahlung photons

**Question3: Elastic scattering of fast neutrons (25 points)**

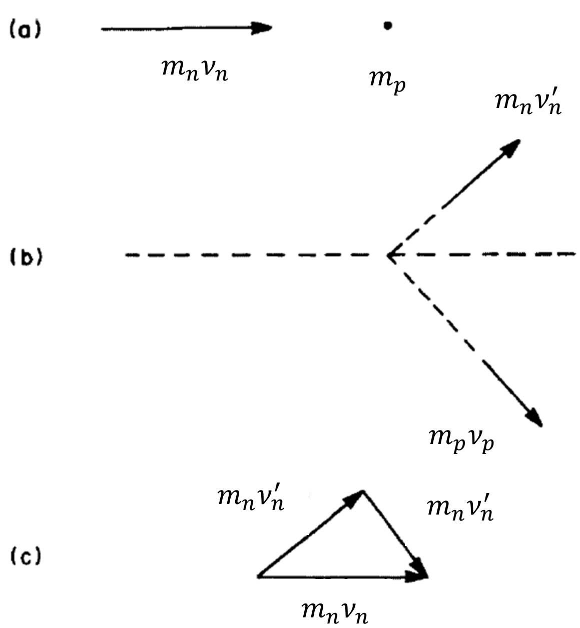
A proton-neutron telescope illustrated below can be used to accurately measure the spectrum of neutrons in a collimated beam. Considering an incident neutron carries a kinetic energy $E_{neutron}$ scattered with a proton, **could you derive the kinetic energy of the recoil proton as a function of the recoil angle $\boldsymbol{\theta}$**?




Arrangement of proton-recoil telescope for measuring spectrum neutron beam.

**Solution:**

We could represent the above neutron-proton scattering problem in the following figures.



From the conservation of energy, we have

$$E_n = E_p + E'_n. \quad (1)$$

Therefore,

$$\frac{1}{2} m_n v_n^2 = \frac{1}{2} m_n v_{n'}^2 + \frac{1}{2} m_p v_p^2. \qquad (2)$$

Considering $m_n \approx m_p$, then

$$v_n^2 = v_{n'}^2 + v_p^2. \qquad (3)$$

Consider the conservation of momentum, we have

$$m_n \overrightarrow{v_n} = m_n \overrightarrow{v'_n} + m_p \overrightarrow{v_p}\,, \qquad (4)$$

and then

$$\overrightarrow{v_n} = \overrightarrow{v'_n} + \overrightarrow{v_p}\,, \qquad (5)$$

or

$$v_{n'}^2 = v_n^2 + v_P^2 - 2 v_n v_p \, cos\, \theta. \qquad (6)$$

Substitute (3) into (6), we have

$$v_n^2 - v_P^2 = v_n^2 + v_P^2 - 2 v_n v_p \, cos\, \theta. \qquad (7)$$

So

$$v_p = v_n \cos\theta, \quad (8)$$

and then the energy of the recoil proton is given by

$$E_p = E_n \cdot cos^2\, \theta.$$

**Question 4: Neutron Activation Analysis (25 points)**

Considering an object containing $n$=$10^{22}$ $^{23}$Na atoms. The object is irradiated by a constant thermal neutrons flux, $\phi$=5000 n/cm$^2$/$s$. The neutrons can potentially be captured by $^{23}$Na atoms and induce the following reaction

$$^{1}_{0}n + ^{23}_{11}Na \rightarrow ^{24}_{11}Na + ^{0}_{0}\gamma$$

The cross-section of $^{23}$Na for thermal neutron is 0.534 barns. $^{24}$Na is subject to radioactive decay with the emission of two gamma rays of energies of 2.75 MeV and 1.37 MeV per disintegration. The half-life of $^{24}$Na is roughly 15 hours. **Please derive the number of $^{24}$Na atoms in the object as a function of time**.

**Solution**:

Considering the generic case that an object is irradiated by a constant flux, $\phi(n/cm^2/s)$, of neutrons, which activates a given types of atoms. Then the net rate of increase of radioactive (daughter) atoms is given by

$$\frac{dN}{dt} = \phi\sigma n - \lambda N, \qquad (1)$$

where $\phi$ = flux, neutrons per cm$^2$ per s,
$\sigma$ = activation cross section, cm$^2$,
$\lambda$ = transformation constant of the induced activity,
$N$ = number of radioactive atoms,
$n$ = number of target atoms.

The above equation is a partial differential equation of the first order of the form

$$\frac{dy}{dt} + P(t)y = Q(t). \qquad (2)$$

The solution for (2) is.

$$y(t) = \frac{\int e^{\int P(t)\cdot dt}\cdot Q(t)\cdot dt + C}{e^{\int P(t)\cdot dt}}. \qquad (3)$$

Therefore, the solution for (1) is given by

$$N(t) = \frac{\int e^{\int \lambda\cdot dt}\cdot \phi\sigma n \cdot dt + C}{e^{\int \lambda\, dt}}$$

$$= \frac{\int e^{\lambda t}\cdot \phi\sigma n \cdot dt + C}{e^{\lambda t}}$$

$$= (\frac{\phi\sigma n}{\lambda} + C\cdot e^{-\lambda t}). \qquad (4)$$

Considering the boundary condition that $N(t = 0) = 0$, then

$$N(t) = \frac{\int e^{\int \lambda \cdot dt} \cdot \phi\sigma n \cdot dt + C}{e^{\int \lambda dt}} = \frac{\phi\sigma n}{\lambda}(1 - e^{-\lambda t}). \qquad (5)$$

From this point, we can substitute the values of the parameters given in the question.