Question 1: Compton scattering of photons (30 points)
Please derive the Compton scattering formula given below (assuming that the electron is at rest when the collision happens, and \( m \) is the rest mass of the electron).

\[
hv' = \frac{hv}{1 + \frac{hv}{mc^2} (1 - \cos \theta)}
\]
Solution:

Conservation of Energy:

\[ h\nu + m_0c^2 = h\nu' + E' \]
\[ h\nu + m_0c^2 = h\nu' + \sqrt{m_0^2c^4 + p'^2c^2} \]

Conservation of Momentum:

x-direction:

\[ \frac{h\nu}{c} = \frac{h\nu'}{c} \cos(\theta) + p'\cos(\phi) \]
\[ p'\cos(\phi) = \frac{h\nu}{c} - \frac{h\nu'}{c} \cos(\theta) \]
\[ \Rightarrow (p'\cos(\phi))^2 = \left(\frac{h\nu}{c}\right)^2 - 2\frac{h\nu}{c} \times \frac{h\nu'}{c} \cos(\theta) + \left(\frac{h\nu'}{c} \cos(\theta)\right)^2 \]

y-direction:

\[ y - \text{dir}: 0 = \frac{h\nu'}{c} \sin(\theta) - p'\sin(\phi) \]
\[ p'\sin(\phi) = \frac{h\nu'}{c} \sin(\theta) \]
\[ \Rightarrow (p'\sin(\phi))^2 = \left(\frac{h\nu'}{c} \sin(\theta)\right)^2 \]

Combining the two equations above:

\[ p'^2 = \left(\frac{h\nu}{c}\right)^2 - 2\frac{h\nu}{c} \times \frac{h\nu'}{c} \cos(\theta) + \left(\frac{h\nu'}{c}\right)^2 \]

Eliminating \(p'^2\) in the energy equation and simplifying: [1 point]

\[ h\nu + m_0c^2 - h\nu' = E' \]
\[ (h\nu)^2 + (m_0c^2)^2 + (h\nu')^2 + 2h\nu m_0c^2 - 2h\nu h\nu' - 2m_0c^2 h\nu' = m_0^2c^4 + p'^2c^2 \]
\[ (h\nu)^2 + (m_0c^2)^2 + (h\nu')^2 + 2h\nu m_0c^2 - 2h\nu h\nu' - 2m_0c^2 h\nu' = m_0^2c^4 + (h\nu)^2 - 2h\nu h\nu' \cos(\theta) + (h\nu')^2 \]
\[ 2h\nu m_0c^2 - 2h\nu h\nu' - 2m_0c^2 h\nu' = -2h\nu h\nu' \cos(\theta) \]
\[ 2h\nu h\nu' \cos(\theta) - 2h\nu h\nu' - 2m_0c^2 h\nu' = -2h\nu m_0c^2 \]
\[ h\nu'(h\nu(1 - \cos(\theta)) + m_0c^2) = h\nu m_0c^2 \]
\[ h\nu' = \frac{h\nu m_0c^2}{h\nu(1 - \cos(\theta)) + m_0c^2} \]
\[ h\nu' = \frac{h\nu}{1 + \frac{h\nu}{m_0c^2}(1 - \cos(\theta))} \]
Question 2: Attenuation coefficient for X-ray and gamma rays in matter (20 points)

What is energy-transfer coefficient? and what is energy-absorption coefficient for X-ray and gamma-rays?

Please explain and write down the equations for these attenuation coefficients and explain the meaning of each individual term in the equations.

Solution:

For a parallel beam of monochromatic gamma rays transmitting through a unit distance in an absorbing material, the energy-transfer coefficient is the fraction of energy that was originally carried by the incident gamma-ray beam and transferred into the kinetic energy of secondary electrons inside the absorber.

The fraction of energy that is carried away by characteristic x-rays following the photoelectric effect. The fraction of energy that is carried away by the two 511keV gamma rays generated by the annihilation of the positron.

\[
\frac{\mu_{tr}}{\rho} = \frac{\tau}{\rho} \left( 1 - \frac{\delta}{h\nu} \right) + \frac{\sigma}{\rho} \left( \frac{E_{avg}}{h\nu} \right) + \frac{\kappa}{\rho} \left( 1 - \frac{2mc^2}{h\nu} \right)
\]

The fraction of energy that is transferred to recoil electron through Compton scattering.

where \( \tau, \sigma, \text{and} \ k \) are the linear attenuation coefficients due to photoelectric effect, Compton scattering, and pair production, respectively. \( \rho \) is the density of the absorbing material.

The energy-absorption coefficient is the fraction of energy that was originally carried by the incident gamma-ray beam and eventually absorbed inside the absorber.

\[
\frac{\mu_{en}}{\rho} = \frac{\mu_{tr}}{\rho} \left( 1 - g \right)
\]

where \( g \) is the average fraction of energy of the initial kinetic energy transferred to electrons that is subsequently emitted as bremsstrahlung photons.
Question 3: Elastic Scattering of Neutrons (40 points)

(a) Please derive the maximum energy that a neutron of mass $M$ and kinetic energy $E_n$ could transfer to a target nucleus of mass $m$ through a single elastic collision.

(b) If a 2.6 MeV neutron has an elastic collision with hydrogen, what is the probability that it loses between 0.63 to 0.75 MeV?

(c) What is the average energy loss by a 2.6 MeV neutron through a single collision with a carbon nucleus?

Hint: The energy loss by neutrons through elastic scattering follows a uniform distribution. To make use of this distribution, you will need to find the lower and upper limits of the distribution first.

Solution:

(a) The above figure shows a neutron (mass $M$ and velocity $V$) approaching a nucleus (mass $m$, at rest). After the collision, which for maximum energy transfer is head-on, the neutron and the recoil nucleus move with speed $V_1$ and $v_1$ along the initial line of travel of the incident neutron. Since the energy and momentum are conserved in the collision, we have the following relationships:

\[
\frac{1}{2}MV^2 = \frac{1}{2}MV_1^2 + \frac{1}{2}mv_1^2
\]

and

\[
MV = MV_1 + mv_1.
\]

Solving the above equations, we obtain

\[
V_1 = \frac{(M - m)V}{M + m}.
\]

Using the above result, we find the maximum energy transfer from the incident neutron to the recoil nucleus given by
\[ Q_{\text{max}} = \frac{1}{2} MV^2 - \frac{1}{2} MV_1^2 = \frac{4mME}{(M + m)^2}, \]

where
\[ E = \frac{MV^2}{2}. \]

(b) The energy transfer from the incident neutron to the recoil nucleus follows a uniform distribution between the 0 and the maximum energy transfer of \( E_{\text{max}} = \frac{4mM}{(v^2 + m)^2} E = E = 2.6 \text{ MeV} \). So the probability of neutron losing energy between 0.63 and 0.75 keV is given by \( p = \frac{0.75 - 0.63}{2.6} = 0.046 \).

(c) The average energy loss by a neutron of 2.6 MeV to a carbon nucleus is
\[ E_{\text{avg energy loss}} = \frac{2Mm}{(M+m)^2} \cdot E_0 = \frac{2 \times 1 \times 2 \times 1}{(12 + 1)^2} = 0.369 \text{ MeV}. \]