NPRE 441: Principles of Radiation Protection

Instructor: Ling-Jian Meng,
Professor,
Department of Nuclear, Plasma and Radiological Engineering,
University of Illinois at Urbana-Champaign
Chapter 1: Radioactivity
Alpha Decay – A Few Learning Objectives

• What is alpha decay?
• How to calculate the energy release from an alpha decay and what is a typical energy spectrum for alpha particles emitted through a given alpha decay process?
• Major health hazards related to alpha emission.
• The indoor Radon problem and its origin.
Alpha Emission

- An alpha particle is a highly energetic helium nucleus consisting of two neutrons and 2 protons.
- It is normally emitted from isotopes when the neutron-to-proton ratio is too low – called the alpha decay.
- Atomic number and atomic mass number are conserved in alpha decays.
Energy Release in Alpha Emission

The required kinetic energy has to come from the decrease in mass following the decay process. The relationship between mass and energy associated with an alpha emission is given as

\[ M_p = M_d + M_\alpha + 2M_e + Q, \]  

(4.1)

where \( M_p \), \( M_d \), \( M_\alpha \), and \( M_e \) are respectively equal to the masses of the parent, the daughter, the emitted alpha particle, and the two orbital electrons that are lost during the transition to the lower atomic numbered daughter, while \( Q \) is the total energy release associated with the radioactive transformation.
Energy Release from Alpha Decay

An example: Alpha decay of $^{226}$Ra

$$^{226}_{88}\text{Ra} \rightarrow ^{222}_{86}\text{Rn} + ^{4}_{2}\text{He}.$$ 

The same example, when considering the daughter atom to have two less electrons,

The energy equation describing $\alpha$ decay is:

$$M_p = M_d + M_\alpha + 2M_e + Q$$
$$Q = M_p - M_d - M_\alpha - 2M_e.$$ 

Here, $M_p$ is the mass of the parent, and $M_d$ is the mass of the progeny, $M_\alpha$ is the mass of the $\alpha$ particle, $M_e$ is the mass of an electron, and $Q$ is the energy released in the reaction. For the $^{226}$Ra example above:

$$Q = 226.025 - 222.0176 - 4.0015 - 2(0.00055)$$
$$Q = 0.00523 \text{ amu} = 4.78 \text{ MeV}.$$ 

What is the energy of the alpha particle?

**FIGURE 3.4.** Nuclear decay scheme of $^{226}_{88}$Ra.
Energy Spectra of Alpha Particles

Radium-226 will decay either with or without an accompanying γ-ray emission. With the γ emission (0.186 MeV, 3.6% of decays), the α particle has an energy of about 4.6 MeV. When there is no γ emission in this case, the α particle has the full energy of 4.78 MeV, and we can look also at the energy of the recoil nucleus from a simple consideration of conservation of energy and momentum:

\[
Q = MV^2/2 + mv^2/2
\]

\[
MV = mv
\]

\[
V = \frac{mv}{M}
\]

\[
Q = \frac{Mm^2v^2}{2M^2} + \frac{mv^2}{2}
\]

\[
E = \frac{mv^2}{2}
\]

\[
Q = E\left(\frac{m}{M} + 1\right)
\]

\[
E = \frac{Q}{1 + m/M}
\]

\[
E = \frac{4.78}{1 + 4/222} = 4.6954 \text{ MeV}
\]

\[
E_{\text{recoil}} \approx 0.088 \text{ MeV}
\]

Measured energy spectrum of alpha particles emitted from the decay of $^{238}\text{Pu}$. 
Alpha Emission and Potential Health Concerns
Radiation Effect and Dose Delivery

For low LET radiation, \( \Rightarrow \text{RBE} \propto \text{LET} \), for higher LET the RBE increases to a maximum, the subsequent drop is caused by the overkill effect.

\[
\text{RBE} = \frac{\text{Dose of } 150 \text{ V X-rays required to cause effect } x}{\text{Dose of radiation required to cause effect } x}
\]

These high energies are sufficient to kill more cells than actually available!
Alpha Emission and Radiation Hazard

(Left) Measured energy spectrum of alpha particles emitted from the decay of $^{238}\text{Pu}$.

In addition to the internal hazard, alpha particles, one can generally expect gamma ray emission with an alpha source. Also, many alpha emitters have radioactive daughters that present radiation protection concerns.
Chapter 1: Radioactivity

Radioactive Decay in Thorium and Uranium Series

THORIUM SERIES

0.3 μsec Po-212
61 min Bi-212
10.6 hr Pb-212
3 min Ti-208

Pb-208 (Stable)

Po-216 (0.15 sec)

Rn-220 (55 sec)

Ra-224 (3.6 day)

Ac-228 (6.1 hr)

Ra-228 (5.8 yr)

Th-232 (1.4 x 10^10 yr)

U-234 (1.17 min)

U-238 (4.5 x 10^9 yr)

URANIUM SERIES

138 day Po-210
160 μsec Po-214
19.7 min Bi-214
3 min Po-218

Pb-206 (Stable)

Po-210 (160 μsec)

Bi-210 (5 day)

Pb-210 (22 yr)

Pb-214 (27 min)

Radon daughters

Rn-222 (3.8 day)

Alpha Decay

Beta Decay

Gamma Emission

http://www.world-nuclear.org/info/inf30.html
Naturally Occurring Radioactivity – Other Isotopes of Radon

All three isotopes of radon has radioactive daughters, so they are all potentially hazardous.

The health concerns of these isotopes are determined by two factors:
• The rate of production from their parent nuclides.
• The probability of decay before get airborne.

\[
\begin{align*}
222_{86}^{\text{Rn}} \text{ (Radon)} & : \text{from } 238_{92}^{\text{U}}, T = 3.81 \text{ days} \\
220_{86}^{\text{Rn}} \text{ (Thoron)} & : \text{from } 232_{90}^{\text{Th}}, T = 56 \text{ seconds} \\
219_{86}^{\text{Rn}} \text{ (Actinon)} & : \text{from } 235_{93}^{\text{U}}, T = 4 \text{ seconds}
\end{align*}
\]

The contributions from the daughters of \(^{220}\text{Rn}\) and \(^{219}\text{Rn}\) to internal exposure are usually negligible compared with that from \(^{222}\text{Rn}\).
Naturally Occurring Radioactivity
– Health Concerns of Radon Gas

Figure 3.11 The $^{226}$Ra decay series.
Beta Decay Processes

• Three favors of beta decay.
• Calculation of the energy release through beta decay and the energy spectrum of beta particles.
• Other processes following beta emission,
  • gamma-ray emission,
  • internal conversion,
  • emission of characteristic X-rays and Auger electrons.
• Master of the complex radiation signatures from a given given beta decay source.
Beta Emission

• Beta particle is an ordinary electron. Many atomic and nuclear processes result in the emission of beta particles.

• One of the most common source of beta particles is the beta decay of nuclides, in which

Beta decay

\[
\frac{A}{Z} X \rightarrow \frac{A}{Z+1} Y + 0_{-1}^1 \beta + \nu
\]

Beta-plus decay

\[
\frac{A}{Z} X \rightarrow \frac{A}{Z-1} Y + 0_{1}^1 \beta + \nu
\]

Electron capture

\[
\frac{A}{Z} X + e^- \rightarrow \frac{A}{Z-1} Y + \nu
\]
Energy Release of Beta Decay

The energy release in a beta decay is given as

\[ Q = M_p - (M_d + M_e) \]

- The energy release is once again given by the conversion of a fraction of the mass into energy. Note that atomic electron bonding energy is neglected.
- For a beta decay to be possible, the energy release has to be positive.
Energy Release of Beta Decay

An example

\[ ^{32}_{15}P \rightarrow ^{32}_{16}S + ^{0}_{-1}e \]

The corresponding energy release is given by

\[ Q = M_p - M_d - M_e = 0.001837 \text{ AMU} \]

or equivalently

\[ Q = 1.71 \text{ MeV} \]

Similar to the case of alpha decay, the energy shared by the recoil nucleus is

\[ \frac{M_e}{(M_p + M_e)} \times Q \]

... So the electron generated will be mono-energetic...
The energy release is shared by all three daughter products. Due to the relatively large mass of the daughter nucleus, it attains only a small fraction of the energy. Therefore, the kinetic energy of the beta particle is

\[ E_{\beta^-} \approx Q - E_{\nu} \]
Examples for Beta Decay

- Beta emissions are normally associated with complicated decay schemes and the emission of other particles such as gamma rays.

- There exist the so called “pure beta emitters”, such as $^3$H, $^{14}$C, $^{32}$P and $^{90}$Sr, which have no accompanying gamma rays.
Energy Release Through Positron Decay

The energy release $Q$ associated with the positron emission process is given by

$$Q \approx M_p - M_d - M_e - M_{e^+} = M_p - (M_d + 2M_e)$$

where the atomic electron binding energy is ignored.
Orbital Electron Capture

In electron capture (EC), one of the atomic electrons is captured by the nucleus and unites with a proton to form a neutron with the emission of a neutron.

\[ \frac{A}{Z} X + e^{-} \rightarrow \frac{A}{Z-1} Y + \nu \]

\[ _{0}^{1}e + _{1}^{1}H \rightarrow _{0}^{1}n + \nu \]

• For neutron deficient atoms to attain stability through positron emission, it must exceed the weight of the daughter by at least two electron masses. If this condition cannot be satisfied, the neutron deficiency can be overcome by the EC process.

• For example,

\[ \frac{103}{46} \text{Pd} + _{-1}^{0}e \rightarrow \frac{103}{45} \text{Rh} + _{0}^{0}\nu. \]
An Example

24. Nuclide A decays into nuclide B by $\beta^+$ emission (24%) or by electron capture (76%). The major radiations, energies (MeV), and frequencies per disintegration are, in the notation of Appendix D:

- $\beta^+$: 1.62 max (16%), 0.98 max (8%)
- $\gamma$: 1.51 (47%), 0.64 (55%), 0.511 (48%, $\gamma^\pm$)
- Daughter X rays
- $e^-$: 0.614

(a) Draw the nuclear decay scheme, labeling type of decay, percentages, and energies.

(b) What leads to the emission of the daughter X rays?
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$$0.614 = E_X - E_B$$
Gamma Ray Emission following Beta Decay
Chapter 1: Radioactivity

Excited Nucleus vs Excited Atom

Diagram showing the transition between an excited nucleus and an excited atom, illustrating the process of electron emission and the formation of a hole in the electron shell.
Internal Conversion

An excited nucleus

De-excite through the emission of a gamma ray

\[ \Rightarrow \text{Gamma Ray Emission} \]

The excitation energy is transferred directly to an orbital electron, causing it to be ejected from the atom

\[ \Rightarrow \text{Internal Conversion} \]

Conversion electron with an energy

\[ E_{\beta^-} = E_{ex} - E_b \]

IC Coefficient (or Branching Ratio)

\[ \frac{N_\gamma}{N_e} \]
Internal Conversion

- Conversion electrons can originate from several different electron shells within the atom, a single excited state generally leads to several groups of electrons with different energies.
- The only practical laboratory scale source of mono-energetic electron groups in high keV to MeV energy range.
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Daughter X rays
$e^-$: 0.614

(a) Draw the nuclear decay scheme, labeling type of decay, percentages, and energies.
(b) What leads to the emission of the daughter X rays?
Chapter 1: Radioactivity

Excited Nucleus vs Excited Atom

Diagram showing the transition of an excited nucleus to an excited atom, with transitions labeled (1), (2), (3), and (4).

- Transition (1): Electron from K shell to L shell
- Transition (2): Emission of an X-ray photon
- Transition (3): Emission of a gamma (\(\gamma\)) photon
- Transition (4): Loss of an electron (\(e^-\)) from the K shell

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 Ionizing Radiation from Excited Atoms
Auger Electrons

- The excitation energy of the atom may be transferred to one of the outer electrons, causing it to be ejected from the atom.
- Auger electrons are roughly the analogue of internal conversion electrons when the excitation energy originates in the atom rather than in the nucleus.

\[ h\nu = E_K - E_L \]

Auger electron, \( E = E_K - E_L - E_M \)

Vacant

Figure 3.7 (A) The usual emission of a K characteristic X-ray, \( h\nu \), energy equal to \( E_K - E_L \), the difference in binding energy for the two orbital electrons, K and L. (B) \( h\nu \) has been absorbed and a monoenergetic Auger electron is emitted, in the example shown, from the M shell, the energy of which is \( E_K - E_L - E_M \). (C) In its final state the atom has vacancies in the L and M orbitals.

\[ E_{a.e.} = (E_K - E_{L_1}) - E_{L_{23}} \]
Auger Electrons

The relative probability of the emission of characteristic radiation to the emission of an Auger electron is called the fluorescent yield, $\omega$:

$$\omega_K = \frac{\text{Number } K \times \text{ray photons emitted}}{\text{Number } K \text{ shell vacancies}} \quad (3-12)$$

Values for $\omega_K$ are given in Table 3-1. We see that for large $Z$ values fluorescent radiation is favored, while for low values of $Z$ Auger electrons tend to be produced.

From this table we see that if a nucleus with $Z = 40$ had a $K$ shell hole, then on the average 0.74 fluorescent photons and 0.26 Auger electrons would be emitted.

<table>
<thead>
<tr>
<th>$Z$</th>
<th>$\omega_K$</th>
<th>$Z$</th>
<th>$\omega_K$</th>
<th>$Z$</th>
<th>$\omega_K$</th>
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<tbody>
<tr>
<td>10</td>
<td>0</td>
<td>40</td>
<td>.74</td>
<td>70</td>
<td>.92</td>
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<td>15</td>
<td>.05</td>
<td>45</td>
<td>.80</td>
<td>75</td>
<td>.93</td>
</tr>
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<td>.19</td>
<td>50</td>
<td>.84</td>
<td>80</td>
<td>.95</td>
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<td>25</td>
<td>.30</td>
<td>55</td>
<td>.88</td>
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<tr>
<td>30</td>
<td>.50</td>
<td>60</td>
<td>.89</td>
<td>90</td>
<td>.97</td>
</tr>
<tr>
<td>35</td>
<td>.63</td>
<td>65</td>
<td>.90</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

From Evans (E1)
Radiation Concerns of Beta Particles

• Energetic beta particles may penetrate the skin and lead to external hazard. In general, beta particles with an energy less than 200keV (such as from $^{35}$S and $^{14}$C) are not considered to be external radiation hazard. If deposited inside the body, beta particles normally lead to a certain degree of radiation exposure.

• Beta emitters may also emit gamma rays that leads to extra radiation exposure. For example, beta-decay of Co-60 leads to gamma emission...

• Beta particles in the MeV range also interact with surrounding materials (especially those contain high Z elements) through bremsstrahlung and therefore induces x-rays. So extra care has to be taken for a proper shielding of an energetic beta source.
Typical Decay Products from Unstable Radioisotopes

**Alpha decay**
- Alpha particles
- Daughter nuclei

**Beta decay**
- Beta decay
  - Beta particles
  - Bremsstrahlung X-rays
- Beta-plus decay
  - Positrons
  - Annihilation gamma rays
- Orbital electron capture (E.C.)
- Daughter nuclei, and
  - Gamma ray emission
    - Gamma rays
  - Excited Daughter Nuclei
  - Internal conversion (I.C.)
    - IC electrons
    - Bremsstrahlung X-rays
    - Auger electrons
    - Characteristic X-rays
    - Excited atoms
Understanding the Radiation from Cs-137

Decay scheme:

\[
{^{137}_{55}}\text{Cs} \rightarrow {^{137}_{56}}\text{Ba} + ^0_1\beta + ^0_0\nu.
\]

What will happen to the excited Ba-137 nucleus?


FIGURE 3.8. Decay scheme of \(^{137}_{55}\)Cs.
If you are holding a Cs-137 source, what are the radiations that your hand/body is exposed to?

\[ ^{137}_{55}\text{Cs} \rightarrow ^{137}_{56}\text{Ba} + ^0_{-1}\beta + ^0_0\bar{\nu}. \]

http://www.nuclear.kth.se/courses/lab/latex/internal/internal.html
Typical Decay Products from Unstable Radioisotopes

Alpha decay
- Alpha particles
- Daughter nuclei

Beta decay
- Beta decay
  - Beta particles
  - Bremsstrahlung X-rays
  - Excited Daughter Nuclei
    - Gamma ray emission
      - Gamma rays
    - Bremsstrahlung X-rays

- Beta-plus decay
  - Positrons
    - Annihilation gamma rays
  - Orbital electron capture (E.C.)
    - Daughter nuclei

- Internal conversion (I.C.)
  - IC electrons
  - Auger electrons
  - Excited atoms
    - Characteristic X-rays
Radioactivity from Direct Excitation X-ray Emission

- Bremsstrahlung emission.
- Energy spectrum of X-rays from a Bremsstrahlung X-ray sources.
X-ray Generation – Bremsstrahlung

- Target nucleus positive charge \((Z \cdot p^+)\) attracts incident \(e^-\)
- Deceleration of an incident \(e^-\) occurs in the proximity of the target atom nucleus
- Energy lost by \(e^-\) is gained by the EM photon (x-ray) generated
  - The impact parameter distance, the closest approach to the nucleus by the \(e^-\) determines the amount of E loss
  - The Coulomb force of attraction varies strongly with distance \((\propto \frac{1}{r^2})\); ↓ distance → ↑ deceleration and E loss → ↑ photon E
  - Direct impact on the nucleus determines the maximum x-ray E \((E_{\text{max}})\)
X-ray Generation – Characteristic X-rays

Superimposed multiple flat spectrum with decreasing cutoff energy

Low energy X-rays suffer attenuation inside the anode

Further attenuation by the source package.

External filtering to reduce low E photons → lower does

Beam hardening

**Figure 5.5**
Relative intensity of x-ray photons. (Adapted from Webster, 1998. This material is used by permission of John Wiley & Sons, Inc.)
Neutron Sources
Cf-252 neutron source can be made extremely compact

An engineer tests the prototype Timed Neutron Detector, a device that detects landmines. The neutron source of the landmine detector holds a tiny amount of californium-252. (Photo credit: Pacific Northwest National Lab)
### Table 7.1 Neutron Terminology

<table>
<thead>
<tr>
<th>Term</th>
<th>Energy Range</th>
<th>Velocity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ultracold</td>
<td>$&lt;2 \times 10^{-7}$ eV</td>
<td>6 m/s</td>
</tr>
<tr>
<td>Very cold</td>
<td>$2 \times 10^{-7}$ eV to $5 \times 10^{-5}$ eV</td>
<td>100 m/s</td>
</tr>
<tr>
<td>Cold neutrons</td>
<td>$5 \times 10^{-5}$ eV to $0.025$ eV</td>
<td>—</td>
</tr>
<tr>
<td>Thermal$^c$</td>
<td>$0.025$ eV</td>
<td>2200 m/s</td>
</tr>
<tr>
<td>Epithermal</td>
<td>$1$ eV–$1$ keV</td>
<td>$4.4 \times 10^5$ m/s</td>
</tr>
<tr>
<td>Cadmium</td>
<td>$&lt;0.4$ eV</td>
<td>8800 m/s</td>
</tr>
<tr>
<td>Epicadmium</td>
<td>$&gt;0.6$ eV</td>
<td>$1.1 \times 10^4$ m/s</td>
</tr>
<tr>
<td>Slow</td>
<td>$&lt;1$ to $10$ eV</td>
<td>$1.4 \times 10^4$ m/s</td>
</tr>
<tr>
<td>Resonance$^a$</td>
<td>1 to $300$ eV</td>
<td>$2.4 \times 10^5$ m/s</td>
</tr>
<tr>
<td>Intermediate</td>
<td>1 keV to 0.1 MeV</td>
<td>$4.4 \times 10^6$ m/s</td>
</tr>
<tr>
<td>Fast</td>
<td>$&gt;0.1$ MeV</td>
<td>$1.4 \times 10^7$ m/s</td>
</tr>
<tr>
<td>Ultra fast (relativistic)</td>
<td>$&gt;20$ MeV</td>
<td>—</td>
</tr>
<tr>
<td>Fission$^b$</td>
<td>100 keV to 15 MeV</td>
<td>—</td>
</tr>
</tbody>
</table>

$^a$In pile neutron physics usually refers to neutrons which are strongly captured in the resonance of U-238, and of a few commonly used detectors, e.g., In, Au.

$^b$Most probable energy 0.8 MeV, Average energy 2.0 MeV.

$^c$Maxwellian distribution of 20°C extends to about 0.1 eV.
Neutron Sources – Spontaneous Fission

Spontaneous fission of transuranic heavy nuclides, such as $^{252}$Cf, produces several fast neutrons, in addition to heavy fission products, prompt fission gamma rays and beta and gamma ray activities.

- Half-life: 2.65 years
- Neutron yield: $0.116 \text{n/s per Bq}$, or $2.3 \times 10^6 \text{n/s per mg}$
- Neutron energy peaking at $0.5 \text{MeV}$ and extends beyond $10 \text{MeV}$.

Measured neutron energy spectrum from spontaneous fission of $^{252}$Cf

Neutron Sources – Radioisotope \((\alpha,n)\) Sources

Energetic alpha particles can induce \((\alpha,n)\) reaction in certain target materials.

\[ \frac{4}{2}\alpha + \frac{9}{4}Be \rightarrow \frac{12}{6}C + \frac{1}{0}n \]  
Q-value: 5.71MeV

The source is normally prepared in the form of alloy \((\text{MBe}_{13})\), where M is alpha-emitting radioisotopes.

A practical neutron source

Neutron Sources – Radioisotope ($\alpha$,n) Sources

A typical neutron energy spectrum from an $^{239}$Pu/Be source.

- The various peak and valley are due to the distinct excited states of the $^{12}\text{C}$ product nucleus.
- The continuum is the result of variable energy possessed by the alpha particles before reaction.
Neutron Sources – Radioisotope ($\alpha$,n) Sources

Radium-226 gamma ray spectrum from high purity germanium (HPGe) detector
Neutron Sources – Photon-Neutron Sources

- Some radioisotope gamma ray emitters can also be used to produce neutrons when combined with an appropriate target material.

\[ ^9\text{Be} + h\nu \rightarrow ^8\text{Be} + ^0\text{n}, \quad Q \text{-value} : -1.666\text{MeV} \]

\[ ^2\text{H} + h\nu \rightarrow ^1\text{H} + ^0\text{n}, \quad Q \text{-value} : -2.226\text{MeV} \]

- A gamma ray photon with an energy greater than the negative of the Q-value is required.

- Some practical gamma ray emitter include: \(^{226}\text{Ra}, ^{124}\text{Sb}, ^{72}\text{Ga}, ^{140}\text{La}\) and \(^{24}\text{Na}\).
Neutron Sources – Photoneutron Sources

If the gamma rays are monoenergetic, the neutrons are also nearly monoenergetic!

\[
E_n(\theta) \approx \frac{M(E_\gamma + Q)}{m + M} + \frac{E_\gamma[(2mM)(m + M)(E_\gamma + Q)]^{1/2}}{(m + M)^2} \cos(\theta)
\]

where
\[\theta = \text{angle between gamma photon and neutron direction}\]
\[E_\gamma = \text{gamma energy}\]
\[M = \text{mass of recoil nucleus} \times c^2\]
\[m = \text{mass of neutron} \times c^2\]

The neutron energy is blurred by
- The slight angular dependency.
- Neutron scattering inside the source.
Neutron Sources – Photoneutron Sources

Calculated neutron energy spectra

Neutron flux

Energy (keV)

Ga–CD₂  Na–CD₂  Na–Be
<table>
<thead>
<tr>
<th>Source</th>
<th>Reaction</th>
<th>Half Life</th>
<th>Average Neutron Energy (MeV)</th>
<th>Yield n/s/Cl</th>
<th>Character Problems</th>
</tr>
</thead>
</table>
| Mock Fission  
(Po + Be) | α,n       | 134.4 d   | Fission spectrum            | 4 × 10⁶      | α                |
| ²⁴¹Na + Be  | γ,n       | 15 h      | 0.83                        | 1.3 × 10⁵     | γ                |
| ²⁴²Na + D₂O | γ,n       | 15 h      | 0.22                        | 2.7 × 10⁴     | γ                |
| ⁵⁶Mn + Be   | γ,n       | 2.58 h    | 0.1 (90%) 0.3 (10%)        | 2.9 × 10⁴     | γ                |
| ⁵⁶Mn + D₂O | γ,n       | 2.58 h    | 0.22                        | 3.1 × 10⁴     | γ                |
| ⁷⁷Ca + Be   | γ,n       | 14.1 h    | 0.78                        | 5 × 10⁴       | γ                |
| ⁷⁷Ga + D₂O | γ,n       | 14.1 h    | 0.13                        | 6 × 10⁴       | γ                |
| ⁸⁸Y + Be    | γ,n       | 107.6 d   | 0.16                        | 1 × 10⁷       | γ                |
| ⁹⁰Y + D     | γ,n       | 107 d     | 0.30                        | 3 × 10⁵       | γ                |
| ¹⁰⁰Ru + Be  | γ,n       | 14 s      | 0.30                        | 8.2 × 10⁴     | γ                |
| ¹²²Sb + Be³ | γ,n       | 60.2 d    | 0.024                       | 1.9 × 10⁴     | γ                |
| ¹⁴⁷La + Be  | γ,n       | 40.3 h    | 0.62                        | 3 × 10⁴       | γ                |
| ¹⁴⁷La + D₂O | γ,n       | 40.3 h    | 0.15                        | 8 × 10³       | γ                |
| ²²⁸Ra + Be  | γ,n       | 5.75 y    | 0.83                        | 3.5 × 10⁴     | γ                |
| ²³¹Ra + D₂O | γ,n       | 5.75 y    | 0.20                        | 9.5 × 10⁴     | γ                |
| ²⁵⁶Ra + Be  | α,n       | 1600 y    | Spectrum                    | 3.0 × 10⁴     | α, γ, Rn          |
| ²⁵⁶Ra + Be  | α,n       | 1600 y    | 5.0                         | 1.7 × 10⁵     | α, γ, Rn          |
| ²⁵⁶Ra + B   | α,n       | 1600 y    | 3.0                         | 6.8 × 10⁵     | α, γ, Rn          |
| ²⁵⁶Ra + D₂O | α,n       | 1600 y    | 0.12                        | 1 × 10⁵       | α, γ, Rn          |
| ²⁵⁸K + Be   | α,n       | 3.82 d    | 5                           | 1.5 × 10⁶     | α, γ, Rn          |
| ²⁵⁸Po + Be  | α,n       | 134.4 d   | 4                           | 3 × 10⁵       | α                |
| ²⁶⁸Po + B   | α,n       | 134.4 d   | 2.5                         | 9 × 10⁵       | α                |
| ²⁶⁸Po + F   | α,n       | 134.4 d   | 1.4                         | 4 × 10⁵       | α                |
| ²⁷⁰Po + Li  | α,n       | 134.4 d   | 0.42                        | 9 × 10⁵       | α                |
| ²⁶⁷Ac + Be  | α,n       | 21.8 y    | —                           | —             | α                |
| ²⁶⁹Pu + Be  | α,n       | 87.7 y    | 4.5                         | 2.3 × 10⁶     | α                |
| ²⁶⁹Pu + Be  | α,n       | 2.41 × 10⁴ y | 4 (3.2) | 1.7 × 10⁸ | α                |
| ²⁷⁰Am + Be  | α,n       | 432 y     | 1.5                         | 2.2 × 10⁶     | α                |
| ²⁷⁰Am + Li  | α,n       | 432 y     | 0.54                        | 6.0 × 10⁴     | α                |
| ²⁷⁰Pu (WG)* | Spont. Fission | 2.41 × 10⁴ y | 1.94 | 63.6 | α                |
| ²⁵⁸Cf      | Spont. Fission | 2.64 y | Fission spectrum³ (2.35) | 10⁶ | α                |

*³80 : 0.035 neutrons per fission.

⁴Typically used in reactors – inserted as ⁰⁹⁵Sb and resultant activation to ⁰⁹⁸Sb occurs.

⁵WG = Weapons grade, >95% Pu-239.
Average Binding Energy Per Nucleon Comparing Fusion and Fission Reactions

FUSION

- fast particles
- deuterium \( + \) \( N \)
- tritium \( + \) \( N \)

\( m = 2 \)
\( m = 3 \)

1 UNIT = energy use of one U.S. citizen in 1 year.

Conversion to energy per kg fuel

\( E = (0.02)c^2 \)
676 units

FISSION

- slow neutron
- \( 235U \)

\( m = 1 \)

one of many possible divisions

\( 90Rb \)
\( 143Cs \)

\( m_{after} = 4.98 \)
\( E = (0.2)c^2 \)
176 units

http://230nsc1.phy-astr.gsu.edu/hbase/hframe.html
Neutrons Generated by Accelerated Charged Particles

- Neutrons can be produced by nuclear reaction between accelerated charged particles.

\[ ^1_2H + ^1_2H \rightarrow ^3_2He + ^0_1n, \quad \text{Q-value: 3.26MeV, } En=2.5\text{MeV} \]

\[ ^2_1H + ^3_1H \rightarrow ^4_2He + ^0_1n, \quad \text{Q-value: 17.6MeV, } En=14.1\text{MeV} \]

**Why accelerated?**

- Due to the coulomb barrier between the incident deuteron and the light target nucleus, only a relatively small accelerating potential is required (about 100 to 300kV) to induce the reaction.

- The neutrons produced by a given nuclear reaction (D-D or D-T) have roughly the same energies.
Chapter 1.2: Transformation Kinetics

- Specific Activity.
- General methodology for the derivation of general serial decay equations (involving 3, 4 or more radioisotopes).
- Secular equilibrium and transient equilibrium, and their practical implications.
Chapter 1: Radioactivity

http://www.world-nuclear.org/info/inf30.html
Serial Transformation

In many situations, the parent nuclides produce one or more radioactive offsprings in a chain. In such cases, it is important to consider the radioactivity from both the parent and the daughter nuclides as a function of time.

\[
{^{90}_{36}}Kr \quad {\beta}^{33 \text{ s}} \rightarrow {^{90}_{37}}Rb \quad {\beta}^{2.74 \text{ min}} \rightarrow {^{90}_{38}}Sr \quad {\beta}^{28.8 \text{ years}} \rightarrow {^{90}_{39}}Y \quad {\beta}^{64.2 \text{ h}} \rightarrow {^{90}_{40}}Zr.
\]

• Due to their short half lives, \(^{90}Kr\) and \(^{90}Rb\) will be completely transformed, results in a rapid building up of \(^{90}Sr\).

• \(^{90}Y\) has a much shorter half-life compared to \(^{90}Sr\). After a certain period of time, the instantaneous amount of \(^{90}Sr\) transformed per unit time will be equal to that of \(^{90}Y\).

• In this case, \(^{90}Y\) is said to be in a secular equilibrium.
Characteristics of Exponential Decay – Average or Mean Life

It is sometimes useful to characterize a radioactive source in terms of the average or mean life of the given isotope, $\tau$. It can be understood as 

\[ \tau = \frac{1}{\lambda} \]

sum of the lifetimes of the individual atoms divided by the total number of atoms originally present.
Specific Activity (SA)

Specific activity of a sample is defined as its activity per unit mass, given in units of Bq/g or Ci/g.

Specific activity for pure radioisotopes is defined as the number of Becquerels per unit mass.

\[
SA = \frac{6.03 \times 10^{23} \text{(atoms/mole)}}{A \text{(g/mole)}} \times \lambda \text{ Bq/g}
\]

SA can be related to the half-life (T) of the radionuclide by

\[
SA = \frac{4.18 \times 10^{23}}{A \cdot T} \text{ Bq/g}
\]
Specific Activity (Continued)

An example:

A solution of Hg(NO₃)₂ tagged with ²⁰³Hg has a specific activity of $1.5 \times 10^5$ Bq/mL ($4 \frac{\mu \text{Ci}}{\text{mL}}$). If the concentration of mercury in the solution is $5 \frac{\text{mg}}{\text{mL}}$,

(a) what is the specific activity of the mercury?

(b) what fraction of the mercury in the Hg(NO₃)₂ is ²⁰³Hg?

(c) what is the specific activity of the Hg(NO₃)₂?
Specific Activity (Continued)

(a) what is the specific activity of the mercury?

Solution:

\[
SA(\text{Hg}) = \frac{\text{activity from Hg per mL}}{\text{weight of Hg per mL}}
\]

\[
= \frac{1.5 \times 10^5 \text{ Bq/mL}}{5 \text{ mg Hg/mL}} = 0.3 \times 10^5 \frac{\text{Bq}}{\text{mg Hg}}.
\]

and the specific activity of \(^{203}\text{Hg}\) is calculated from

\[
SA = \frac{4.18 \times 10^{23}}{A \cdot T} \frac{\text{Bq}}{\text{g}} = \frac{4.18 \times 10^{23}}{203 \cdot 46.5 \text{d} \cdot 24 \text{h} \cdot 3600 \text{s/h}} \frac{\text{Bq}}{\text{g}} = 5.2 \times 10^{14} \frac{\text{Bq}}{\text{g}}.
\]
Specific Activity (Continued)

(b) what fraction of the mercury in the Hg(NO₃)₂ is $^{203}$Hg?

Solution:

The weight-fraction of mercury that is tagged is given by $\frac{SA(\text{Hg})}{SA(^{203}\text{Hg})}$, and the specific activity of $^{203}$Hg is calculated from

$$SA = \frac{4.18 \times 10^{23}}{A \cdot T} \text{ Bq/ g} = \frac{4.18 \times 10^{23}}{203 \cdot 46.5 \text{ d} \cdot 24 \text{ h} / d \cdot 3600 \text{ s} / h} \text{ Bq/ g} = 5.2 \times 10^{14} \text{ Bq/ g}$$

The weight fraction of $^{203}$Hg, therefore, is

$$\frac{SA(\text{Hg})}{SA(^{203}\text{Hg})} = \frac{0.3 \times 10^8 \text{ Bq/ g Hg}}{5.2 \times 10^{14} \text{ Bq/ g }^{203}\text{Hg}} = 5.8 \times 10^{-8} \frac{\text{g }^{203}\text{Hg}}{\text{g Hg}}.$$
Specific Activity (Continued)

(c) what is the specific activity of the Hg(NO₃)₂?

Solution:

Since an infinitesimally small fraction of the mercury is tagged with ²⁰³Hg, it may be assumed that the formula weight of the tagged Hg (NO₃)₂ is 324.63 and that the concentration of Hg (NO₃)₂ is

\[
\frac{324.63 \text{ mg Hg} \text{ (NO₃)₂}}{200.61 \text{ mg Hg}} \times \frac{5 \text{ mg Hg}}{\text{ mL}} = 8.1 \frac{\text{ mg Hg (NO₃)₂}}{\text{ mL}}.
\]

The specific activity,

\[
\frac{1.5 \times 10^5 \text{ Bq/mL}}{8.1 \text{ mg Hg (NO₃)₂/mL}} = 1.9 \times 10^4 \frac{\text{ Bq}}{\text{ mg Hg (NO₃)₂}} \left[ 0.5 \frac{\mu\text{Ci}}{\text{mg Hg (NO₃)₂}} \right].
\]
Naturally Occurring Radioactivity – Health Concerns of Radon Gas

Figure 3.11  The $^{226}$Ra decay series.
General Case

Consider a more general case, in which (a) the half-life of the parent can be of any conceivable value and (b) no restrictions are applied on the relative half-lives of both the parent and the daughter.

\[ A \xrightarrow{\lambda_A} B \xrightarrow{\lambda_B} C, \]

The number of atoms of the parent \( A \) and the daughter \( B \) at any given time \( t \) are therefore related by

\[
N_B = \frac{\lambda_A N_{A0}}{\lambda_B - \lambda_A} \left( e^{-\lambda_A t} - e^{-\lambda_B t} \right)
\]
Proof of the Previous Serial Decay Equation
From Cember, p123-124

of the daughter, it follows that secular equilibrium is a special case of a more general situation in which the half-life of the parent may be of any conceivable magnitude, but greater than that of the daughter. For this general case, where the parent activity is not relatively constant,

\[ A \xrightarrow{\lambda_A} B \xrightarrow{\lambda_B} C, \]

the time rate of change of the number of atoms of species \( B \) is given by the differential equation

\[ \frac{dN_B}{dt} = \lambda_A N_A - \lambda_B N_B. \tag{4.42} \]

In this equation, \( \lambda_A N_A \) is the rate of transformation of species \( A \) and is exactly equal to the rate of formation of species \( B \), the rate of transformation of isotope \( B \) is \( \lambda_B N_B \), and the difference between these two rates at any time is the instantaneous rate of growth of species \( B \) at that time.

According to Eq. (4.18), the value of \( \lambda_A \) in Eq. (4.42) may be written as

\[ N_A = N_{A_0} e^{-\lambda t}. \tag{4.43} \]

Equation (4.42) may be rewritten, after substituting the expression above for \( N_A \) and transposing \( \lambda_B N_B \), as

\[ \frac{dN_B}{dt} + \lambda_B N_B = \lambda_A N_{A_0} e^{-\lambda t}. \tag{4.44} \]
Proof of The Serial Decay Equation (Continued)

\[ \frac{dN_B}{dt} + \lambda_B N_B = \lambda_A N_A e^{-\lambda_A t}, \]  
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Proof of The Serial Decay Equation (Continued)

\[ N_B e^{-\lambda_B t} = \int \lambda_A N_A e^{(\lambda_B - \lambda_A) t} \, dt + C. \quad (4.48) \]

If the integrand in Eq. (4.48) is multiplied by the integrating factor \( \lambda_B - \lambda_A \), then Eq. (4.48) is in the form

\[ \int e^v du = e^v + C \quad (4.49) \]

and the solution is

\[ N_B e^{\lambda_B t} = \frac{1}{\lambda_B - \lambda_A} \lambda_A N_A e^{(\lambda_B - \lambda_A) t} + C. \quad (4.50) \]

If \( t = 0, N_B = 0 \), then

\[ N_B = \frac{\lambda_A N_A}{\lambda_B - \lambda_A} \left( e^{-\lambda_A t} - e^{-\lambda_B t} \right) \]
Quiz 2, Question 1

Question 1: Consider a decay chain, $A \xrightarrow{\lambda_A} B \xrightarrow{\lambda_B} C \xrightarrow{\lambda_C} D$. With the following boundary condition, $N_A(t = 0) = N_{A_0}$, $N_B(t = 0) = 0$, and $N_C(t = 0) = 0$. Please derive the number of Atom C as a function of time.

Solution:

The number of atom C at time $t$ can be determined by the following equation

$$\frac{dN_C}{dt} = \lambda_B N_B - \lambda_C N_C. \hspace{1cm} (1)$$

From the previous solution for the three-element decay chain, we have

$$\lambda_B N_B = \frac{\lambda_B \lambda_A N_{A_0}}{\lambda_B - \lambda_A} (e^{-\lambda_A t} - e^{-\lambda_B t}). \hspace{1cm} (2)$$

Rearranging the terms in (2),

$$\frac{dN_C}{dt} + \lambda_C N_C = \frac{\lambda_B \lambda_A N_{A_0}}{\lambda_B - \lambda_A} (e^{-\lambda_A t} - e^{-\lambda_B t}). \hspace{1cm} (3)$$

Note that the solution for the following linear differential equation

$$\frac{dy}{dx} + P(x) \cdot y = Q(x),$$

is given by

$$y = \frac{\int e^{\int P(x) \, dx} Q(x) \, dx + H}{e^{\int P(x) \, dx}}, \hspace{1cm} (4)$$

where $H$ is a constant that can be determined with boundary conditions.
Activities from the Parent and the Daughter

\[ Q_B = \lambda_B N_B = \frac{\lambda_B \lambda_A N_{A_0}}{\lambda_B - \lambda_A} \left( e^{-\lambda_A t} - e^{-\lambda_B t} \right) \]

\[ Q_A = N_A \cdot \lambda_A = \lambda_A \cdot N_{A_0} \cdot e^{-\lambda_A t} \]
Activity Peaking Times Under General Case

\[ Q_B = \frac{\lambda_B N_B}{\lambda_B - \lambda_A} \left( e^{-\lambda_A t} - e^{-\lambda_B t} \right) \]

\[ Q_A = N_A \cdot \lambda_A = \lambda_A \cdot N_A e^{-\lambda_A t} \]
Activity Peaking Time Under General Case

The peak-reaching-time for the activity from the daughter can be derived as the following:

Start from the equation for the general case

\[
\lambda_B N_B = \frac{\lambda_B \lambda_A N_{A_0}}{\lambda_B - \lambda_A} (e^{-\lambda_A t} - e^{-\lambda_B t})
\]

Differentiate respect to \( t \) and set to zero

\[
\frac{d(\lambda_B N_B)}{dt} = \frac{\lambda_B \lambda_A N_{A_0}}{\lambda_B - \lambda_A} (-\lambda_A e^{-\lambda_A t} + \lambda_B e^{-\lambda_B t}) = 0,
\]

\[
\lambda_A e^{-\lambda_A t} = \lambda_B e^{-\lambda_B t}
\]

and therefore

\[
\ln \frac{\lambda_B}{\lambda_A} = (\lambda_B - \lambda_A) t
\]

\[
t = t_{md} = \frac{\ln(\lambda_B/\lambda_A)}{\lambda_B - \lambda_A} = \frac{2.3 \log(\lambda_B/\lambda_A)}{\lambda_B - \lambda_A}.
\]
Several Special Cases
Secular Equilibrium: $T_A >> T_B (\lambda_A << \lambda_B)$

From this relationship,

$$N_B = \frac{\lambda_A N_A}{\lambda_B} (1 - e^{-\lambda_B t})$$

one can see that

1. As the time goes by, $e^{-\lambda}$ decreases and $Q_B$ approaches $Q_A$.
2. Since $A$ has a relatively long half life, $Q_A$ may be considered as a constant. So the total activity converges to a constant.
Activity Peaking Times Under General Case

From "<Radiation Protection and Dosimetry>>", by Michael Stabin.

Figure 3.11 The $^{226}$Ra decay series.

We can continue on with a species D, E, F, and so on, but the relationships among the species obviously become more complicated and are difficult to categorize. If Species A is very long-lived, however, relative to other members of the chain, after a long time (seven to ten half-lives of the longest-lived progeny species), all the members of the chain will be in secular equilibrium and decaying with the half-life of Species A, and all having the same activity as Species A. An important example is the $^{226}$Ra decay series (Figure 3.11).
Transient Equilibrium: $T_A \geq T_B \ (\lambda_A \leq \lambda_B)$ and $t > t_{md}$

General case

$$\lambda_B N_B = \frac{\lambda_B \lambda_A N_{A0}}{\lambda_B - \lambda_A} \left(e^{-\lambda_A t} - e^{-\lambda_B t}\right)$$

Transient Equilibrium

$$\lambda_B N_B = \frac{\lambda_B \lambda_A N_A}{\lambda_B - \lambda_A}$$

$$Q_B = \frac{\lambda_B}{\lambda_B - \lambda_A} Q_A$$
Summary of Serial Transformations

\[ A \xrightarrow{\lambda_A} B \xrightarrow{\lambda_B} C, \]

General case

Secular Equilibrium

Transient Equilibrium

No Equilibrium

- **T_A > T_B**
  - \[ T_A >> T_B, \quad t > 7T_B \]
  - \[ N_B = \frac{\lambda_A N_A}{\lambda_B - \lambda_A} (e^{-\lambda_A t} - e^{-\lambda_B t}) \]

- **T_A \geq T_B**
  - \[ t > T_{md} \]
  - \[ Q_B = Q_A (1 - e^{-\lambda_B t}) \]

- **T_A < T_B**
  - \[ Q_B = \frac{\lambda_B}{\lambda_B - \lambda_A} Q_A \]
  - ---

---

NPRE 441, Principles of Radiation Protection
An Example

A sample contains 1 mCi of $^{191}\text{Os}$ at time $t = 0$. The isotope decays by $\beta^-$ emission into metastable $^{191m}\text{Ir}$, which then decays by $\gamma$ emission into $^{191}\text{Ir}$. The decay and half-lives can be represented by writing

$$^{191}_{76}\text{Os} \xrightarrow{\beta^-} ^{191m}_{77}\text{Ir} \xrightarrow{\gamma} ^{191}_{77}\text{Ir}.$$  

(c) How many atoms of $^{191m}\text{Ir}$ decay between $t = 100$ s and $t = 102$ s?

(d) How many atoms of $^{191m}\text{Ir}$ decay between $t = 30$ d and $t = 40$ d?
Secular Equilibrium: $T_A >> T_B$

(c) How many atoms of $^{191m}\text{Ir}$ decay between $t = 100$ s and $t = 102$ s?

\[
\begin{array}{c}
^{191}_{76}\text{Os} \xrightarrow{\beta^-} ^{191m}_{77}\text{Ir} \xrightarrow{\gamma} ^{191}_{77}\text{Ir}.
\end{array}
\]

Since (i) $T_A >> T_B$, and (ii) $t=100-102$ s is longer than 7 times $T_B$, we are looking at a secular equilibrium ...

Therefore, the activity from $^{191m}\text{Ir}$ is roughly equal to the activity from a constant number of $^{191}\text{Os}$.

\[
\lambda_A N_A = \lambda_B N_B \quad \text{and} \quad Q_A = Q_B
\]
Secular Equilibrium: $T_A >> T_B (\lambda_A << \lambda_B)$

From this relationship,

$$N_B = \frac{\lambda_A N_A}{\lambda_B} (1 - e^{-\lambda_B t}).$$

$$Q_B = Q_A (1 - e^{-\lambda_B t}),$$

one can see that

1. As the time goes by, $e^{-\lambda_B t}$ decreases and $Q_B$ approaches $Q_A$.

At equilibrium, we have

$$\lambda_A N_A = \lambda_B N_B \text{ and } Q_A = Q_B$$

2. Since $A$ has a relatively long half life, $Q_A$ may be considered as a constant. So the total activity converges to a constant.
Secular Equilibrium: $T_A \gg T_B$

$^{191}\text{Os} \xrightarrow{\beta^-} \frac{15.4}{15.4 \text{ d}} 191^m\text{Ir} \xrightarrow{\gamma} \frac{4.94}{4.94 \text{ s}} 191\text{Ir}.$

Since $Q_{\text{Os}} = 1 \text{ mCi}$, then $Q_{\text{Ir}} \approx 1 \text{ mCi}$.

Therefore, the number of $^{191m}\text{Ir}$ decayed between 100s and 102s is

$$2 \times 3.7 \times 10^7 = 7.4 \times 10^7.$$
A sample contains 1 mCi of $^{191}$Os at time $t = 0$. The isotope decays by $\beta^-$ emission into metastable $^{191m}$Ir, which then decays by $\gamma$ emission into $^{191}$Ir. The decay and half-lives can be represented by writing

$$^{191}_{76}\text{Os} \xrightarrow{\beta^-}^{191m}_{77}\text{Ir} \xrightarrow{\gamma}^{191}_{77}\text{Ir}.$$  

(d) How many atoms of $^{191m}$Ir decay between $t = 30$ d and $t = 40$ d?

**Solution:**

This part is like (c), except that the activities $A_1$ and $A_2$ do not stay constant during the time between 30 and 40 d. Since transient equilibrium exists, the numbers of atoms of $^{191m}$Ir and $^{191}$Os that decay are equal. The number of $^{191m}$Ir atoms that decay, therefore, is equal to the integral of the $^{191}$Os activity during the specified time ($t$ in days):

$$3.7 \times 10^7 \int_{30}^{40} e^{-0.693t/15.4} \, dt = \frac{3.7 \times 10^7}{-0.0450} \left[ e^{-0.0450t} \right]_{30}^{40}$$

$$= -8.22 \times 10^8(0.165 - 0.259)$$

$$= 7.73 \times 10^7.$$
Chapter 2: Interaction of Radiation with Matter
4.1 Interaction of Beta Particles
Key Aspects of Beta Interactions

Collisional interactions of beta particles with matter.
Specific energy loss of beta particles.
Mass stopping, what and why?
Radiative energy loss of beta particles.
Relative importance of collisional and radiative energy loss.
Fraction of energy loss due to Bremsstrahlung process and its implementation to shielding design for beta particles.
Range of beta particles.
Backscattering of beta particles.
Mechanisms of Energy Loss by Electrons

**Ionization and excitation:**
Beta particles may interact with orbital electrons through the electric fields surrounding these charged particles, which leads to excitation and ionization.

Ionization process can be modeled as an inelastic collision, the energy loss by the electron and the kinetic energy carried by the ejected electron is related by

\[ E_k = E_{loss} - \phi \]

where \( \phi \) is the ionization potential of the absorbing medium.
Specific Energy Loss of Beta Particles

Specific energy loss: the linear rate of energy loss by an electron through excitation and ionization, which is given by

\[
\frac{dE}{dx} = \frac{2\pi q^4 N Z \times (3 \times 10^9)^4}{E_m \beta^2 (1.6 \times 10^{-6})^2} \left\{ \ln \left[ \frac{E_m E_k \beta^2}{I^2 (1 - \beta^2)} \right] - \beta^2 \right\} \text{MeV/cm}
\]

where 
- \( q \) = charge on the electron, 1.6 \times 10^{-19} \text{C},
- \( N \) = number of absorber atoms per cm³,
- \( Z \) = atomic number of the absorber,
- \( NZ \) = number of absorber electrons per cm³ = 3.88 \times 10^{20} \text{ for air at } 0^\circ \text{ and } 76 \text{ cm Hg},
- \( E_m \) = energy equivalent of electron mass, 0.51 MeV,
- \( E_k \) = kinetic energy of the beta particle, MeV,
- \( \beta \) = v/c,
- \( I \) = mean ionization and excitation potential of absorbing atoms, MeV,
- \( I = 8.6 \times 10^{-5} \text{ for air; for other substances, } I = 1.35 \times 10^{-5} Z. \)
Mechanisms of Energy Loss

Energy expenditure for creating ion pairs in media:

The average energy needed for creating an ion pair is normally 2 to 3 times greater than the corresponding electron binding energy in the absorbing medium.

<table>
<thead>
<tr>
<th>Gas</th>
<th>Ionization potential</th>
<th>Mean energy expenditure per ion pair</th>
</tr>
</thead>
<tbody>
<tr>
<td>H₂</td>
<td>13.6 eV</td>
<td>36.6 eV</td>
</tr>
<tr>
<td>He</td>
<td>24.5</td>
<td>41.5</td>
</tr>
<tr>
<td>N₂</td>
<td>14.5</td>
<td>34.6</td>
</tr>
<tr>
<td>O₂</td>
<td>13.6</td>
<td>30.8</td>
</tr>
<tr>
<td>Ne</td>
<td>21.5</td>
<td>36.2</td>
</tr>
<tr>
<td>A</td>
<td>15.7</td>
<td>26.2</td>
</tr>
<tr>
<td>Kr</td>
<td>14.0</td>
<td>24.3</td>
</tr>
<tr>
<td>Xe</td>
<td>12.1</td>
<td>21.9</td>
</tr>
<tr>
<td>Air</td>
<td></td>
<td>33.7</td>
</tr>
<tr>
<td>CO₂</td>
<td>14.4</td>
<td>32.9</td>
</tr>
<tr>
<td>CH₄</td>
<td>14.5</td>
<td>27.3</td>
</tr>
<tr>
<td>C₂H₂</td>
<td>11.6</td>
<td>25.7</td>
</tr>
<tr>
<td>C₂H₄</td>
<td>12.2</td>
<td>26.3</td>
</tr>
<tr>
<td>C₆H₆</td>
<td>12.8</td>
<td>24.6</td>
</tr>
</tbody>
</table>

The deviation between the ionization energy and the average energy required to create an ion pair is due to the excitation of the atoms, which does not lead to ionization.

Cember, Introduction to Health Physics, Fourth Edition
Mass Stopping Power

It is also common to specify the energy loss of beta particles in a medium in terms of mass stopping power, which is given by

\[ S = \frac{\text{specific energy loss}(MeV/cm)}{\text{density}(g/cm^3)} = \frac{dE/dx}{\rho}(MeV \cdot cm^2/g) \]

where \( \rho \) is the density of the absorbing medium.

In health physics, it is sometimes important to show the mass stopping power of different absorbers relative to that of air – the relative mass stopping power

\[ \rho_m = \frac{S_{\text{medium}}}{S_{\text{air}}} \approx \frac{S_{\text{medium}}}{3.67} \left( \frac{MeV}{g/cm^2} \right) \]

Why mass stopping power?
Key Aspects of Beta Interactions

Collisional interactions of beta particles with matter.

Specific energy loss of beta particles.

Mass stopping, what and why?

Radiative energy loss of beta particles.

Relative importance of collisional and radiative energy loss.

Fraction of energy loss due to Bremsstrahlung process and its implementation to shielding design for beta particles.

Range of beta particles.

Backscattering of beta particles.
Radiative Energy Loss of Beta Particles – Bremsstrahlung

*Bremsstrahlung* occurs when a beta particle is deflected or accelerated in the forced field of nucleus.
Radiative Energy Loss of Beta Particles – Bremsstrahlung

Part of the energy possessed by the beta particle is emitted in the form of photons. The rate of energy loss is proportional to the square of the instantaneous acceleration experienced by the beta particle.

\[- \left( \frac{dE}{dx} \right)_r = \frac{NEZ(Z + 1)e^4}{137m_0^2c^4} \left( 4 \ln \frac{2E}{m_0c^2} - \frac{4}{3} \right)\]
Radiative Energy Loss of Beta Particles – Bremsstrahlung

Characteristics of Bremsstrahlung Process

**Bremsstrahlung** process becomes increasingly important at higher energy, say in the MeV range.

The efficiency of bremsstrahlung in elements **varies nearly as** \( Z^2 \) (In comparison, the energy loss due to ionization and excitation is proportional to \( Z \)).

In MeV energy range, the rate of energy loss through bremsstrahlung **increases nearly linearly with beta energy**, whereas \((-dE/dx)\) by ionization and excitation increases only with the logarithm of beta energy.

The ratio between the energy loss due to ionization-excitation and **bremsstrahlung** is approximately given by

\[
\frac{(-dE/dx)_{\text{bremsstrahlung}}}{(-dE/dx)_{\text{ionization–excitation}}} \approx \frac{ZE_\beta (MeV)}{800}
\]
Characteristics of Bremsstrahlung

The total linear energy loss of beta particles is given by

\[
(- \frac{dE}{dx})_{\text{total}} = (- \frac{dE}{dx})_{\text{bremsstrahlung}} + (- \frac{dE}{dx})_{\text{ionization–excitation}}
\]

\[
\text{(Total)} \quad \text{(Bremsstrahlung)} \quad \text{(Ionization)}
\]

\[
\sim 3mc^2 \quad E_c \quad E
\]

\( E_c \) is called the critical energy
# Radiative Energy Loss of Beta Particles – Bremsstrahlung

## Table 6.1: Electron Collisonal, Radiative, and Total Mass Stopping Powers; and Range in Water

<table>
<thead>
<tr>
<th>Kinetic Energy</th>
<th>$\beta^2$</th>
<th>$\frac{1}{\rho} \frac{dE}{dx}$</th>
<th>$\frac{-1}{\rho} \frac{dE}{dx}$</th>
<th>$\frac{-1}{\rho} \frac{dE}{dx}$</th>
<th>Rad</th>
</tr>
</thead>
<tbody>
<tr>
<td>10 eV</td>
<td>0.00004</td>
<td>4.0</td>
<td>—</td>
<td>4.0</td>
<td></td>
</tr>
<tr>
<td>30</td>
<td>0.00012</td>
<td>44.</td>
<td>—</td>
<td>44.</td>
<td></td>
</tr>
<tr>
<td>50</td>
<td>0.00020</td>
<td>170.</td>
<td>—</td>
<td>170.</td>
<td></td>
</tr>
<tr>
<td>75</td>
<td>0.00029</td>
<td>272.</td>
<td>—</td>
<td>272.</td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>0.00039</td>
<td>314.</td>
<td>—</td>
<td>314.</td>
<td></td>
</tr>
<tr>
<td>200</td>
<td>0.00078</td>
<td>298.</td>
<td>—</td>
<td>298.</td>
<td></td>
</tr>
<tr>
<td>500 eV</td>
<td>0.00195</td>
<td>194.</td>
<td>—</td>
<td>194.</td>
<td></td>
</tr>
<tr>
<td>1 keV</td>
<td>0.00390</td>
<td>126.</td>
<td>—</td>
<td>126.</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.00778</td>
<td>77.5</td>
<td>—</td>
<td>77.5</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.0193</td>
<td>42.6</td>
<td>—</td>
<td>42.6</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0.0380</td>
<td>23.2</td>
<td>—</td>
<td>23.2</td>
<td></td>
</tr>
<tr>
<td>25</td>
<td>0.0911</td>
<td>11.4</td>
<td>—</td>
<td>11.4</td>
<td></td>
</tr>
<tr>
<td>50</td>
<td>0.170</td>
<td>6.75</td>
<td>—</td>
<td>6.75</td>
<td></td>
</tr>
<tr>
<td>75</td>
<td>0.239</td>
<td>5.08</td>
<td>—</td>
<td>5.08</td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>0.301</td>
<td>4.20</td>
<td>—</td>
<td>4.20</td>
<td></td>
</tr>
<tr>
<td>200</td>
<td>0.483</td>
<td>2.84</td>
<td>0.006</td>
<td>2.85</td>
<td></td>
</tr>
<tr>
<td>500</td>
<td>0.745</td>
<td>2.06</td>
<td>0.010</td>
<td>2.07</td>
<td></td>
</tr>
<tr>
<td>700 keV</td>
<td>0.822</td>
<td>1.94</td>
<td>0.013</td>
<td>1.95</td>
<td></td>
</tr>
<tr>
<td>1 MeV</td>
<td>0.886</td>
<td>1.87</td>
<td>0.017</td>
<td>1.89</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.987</td>
<td>1.91</td>
<td>0.065</td>
<td>1.98</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>0.991</td>
<td>1.93</td>
<td>0.084</td>
<td>2.02</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0.998</td>
<td>2.00</td>
<td>0.183</td>
<td>2.18</td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>0.999+</td>
<td>2.20</td>
<td>2.40</td>
<td>4.60</td>
<td></td>
</tr>
<tr>
<td>1000 MeV</td>
<td>0.999+</td>
<td>2.40</td>
<td>26.3</td>
<td>28.7</td>
<td></td>
</tr>
</tbody>
</table>

*From Atoms, Radiation, and Radiation Protection, James E Turner, p140*
Energy Loss by Bremsstrahlung

For beta particles to stop completely, the fraction of energy loss by Bremsstrahlung process is approximately given by

\[ f_\beta = 3.5 \times 10^{-4} Z E_m, \]  \hspace{1cm} (5.1)

where
- \( f_\beta \) = the fraction of the incident beta energy converted into p
- \( Z \) = atomic number of the absorber,
- \( E_m \) = maximum energy of the beta particle, MeV.
Energy Loss by Bremsstrahlung

An example

A very small source (physically) of $3.7 \times 10^{10}$ Bq (1 Ci) of $^{32}$P is inside a lead shield just thick enough to prevent any beta particles from emerging. What is the bremsstrahlung energy flux at a distance of 10 cm from the source (neglect attenuation of the bremsstrahlung by the beta shield)?

Solution:

The fraction of energy emitted in the form of bremsstrahlung is

$$f_\beta = 3.5 \times 10^{-4} \quad ZEM = 3.5 \times 10^{-4} \times 82 \times 1.71 = 0.049.$$

The total amount of kinetic energy carried by the electrons emitted by the source is

$$E_\beta \text{ (MeV/s)} = \frac{1}{3} \frac{E_{\text{max}} \text{ MeV}}{\beta} \times 3.7 \times 10^{10} \frac{\beta}{\text{s}}$$
Energy Loss by Bremsstrahlung

An example (continued)

For health physics purposes, it is assumed that all the bremsstrahlung photons are of the beta particle’s maximum energy, $E_{\text{max}}$. The photon flux $\phi$ of bremsstrahlung photons at a distance $r$ cm from a point source of beta particles whose activity is $3.7 \times 10^{10}$ Bq (1 Ci) is therefore given as

$$\phi = \frac{f E_\beta}{4\pi r^2 E_{\text{max}}}$$

$$= \frac{0.049 \times \frac{1}{3} \times 1.71 \frac{\text{MeV}}{\beta} \times 3.7 \times 10^{10} \frac{\beta}{\text{s}}}{4\pi \times (10 \text{ cm})^2 \times 1.71 \text{ MeV/photon}} = 4.8 \times 10^5 \frac{\text{photons/s}}{\text{cm}^2}.$$
Backscattering

The fact that electrons often undergo large-angle deflections along their tracks leads to the phenomenon of backscattering. An electron entering one surface of an absorber may undergo sufficient deflection so that it re-emerges from the surface through which it entered. These backscattered electrons do not deposit all their energy in the absorbing medium and therefore can have a significant effect on the response of detectors designed to measure the energy of externally incident electrons. Electrons that backscatter in the detector “entrance window” or dead layer will escape detection entirely.

Knoll, Radiation Detection and measurements, p47.
Monte Carlo Simulation of Electron Paths. This simulation is of 15 KeV electrons in fayalite (Fe$_2$SiO$_4$). Distances are given in nanometers (1000 nm = 1 µm). Paths of backscattered electrons are in red; those of absorbed electrons in blue. One should remember that this slice through a three-dimensional volume. This model was run using the Casino software described at http://www.gel.usherbrooke.ca/casino/What.html.

http://www4.nau.edu/microanalysis/Microprobe-SEM/Signals.html
Backscattering

Backscattering is most pronounced for electrons with low incident energy and absorbers with high atomic number. Figure 2.17 shows the fraction $\eta$ of monoenergetic electrons that are backscattered from thick slabs of various materials, as a function of incident energy $E$. (From Tabata et al.$^{27}$)

Figure 2.17 Fraction $\eta$ of normally incident electrons that are backscattered from thick slabs of various materials, as a function of incident energy $E$. (From Tabata et al.$^{27}$)
4.2 Interaction of Heavy Charged Particles

Understanding of the linear stopping power of a given media for heavy charged particles.

Beth formula –

  Implications,

  Implementation and

  Limitations.
Energy Loss Mechanisms

Heavy charged particles loss energy primarily though the ionization and excitation of atoms.

Heavy charged particles can transfer only a small fraction of its energy in a single collision. Its deflection in collision is almost negligible. Therefore heavy charged particles travel in almost straight paths in matter, losing energy continuously through a large number of collisions with atomic electrons.

At low velocity, a heavy charged particle may losses a negligible amount of energy in nuclear collisions. It may also pick up free electrons along its path, which reduces its net charge.
Energy Loss Mechanisms

For heavy charged particles, the maximum energy that can be transferred in a single collision is given by the conservation of energy and momentum:

\[
\frac{1}{2} M V^2 = \frac{1}{2} M V_1^2 + \frac{1}{2} m v_1^2
\]

\[
M V = M V_1 + m v_1.
\]

where \( M \) and \( m \) are the mass of the heavy charged particle and the electron. \( V \) is the initial velocity of the charged particle. \( V_1 \) and \( v_1 \) are the velocities of both particles after the collision.

The maximum energy transfer is therefore given by

\[
Q_{\text{max}} = \frac{1}{2} M V^2 - \frac{1}{2} M V_1^2 = \frac{4mME}{(M + m)^2}
\]
Maximum Energy Loss by a Single Collision

For a more general case, which includes the relativistic effect, the maximum energy transferred by a single collision is

\[ Q_{\text{max}} = \frac{2\gamma^2 mV^2}{1 + 2\gamma m/M + m^2/M^2} \]

where \( \gamma = 1/\sqrt{1 - \beta^2} \), \( \beta = V/c \), and \( c \) is the speed of light.
Maximum Energy Loss by a Single Collision

**TABLE 5.1. Maximum Possible Energy Transfer, $Q_{\text{max}}$, in Proton Collision with Electron**

<table>
<thead>
<tr>
<th>Proton Kinetic Energy $E$ (MeV)</th>
<th>$Q_{\text{max}}$ (MeV)</th>
<th>Maximum Percentage Energy Transfer $100Q_{\text{max}}/E$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.00022</td>
<td>0.22</td>
</tr>
<tr>
<td>1</td>
<td>0.0022</td>
<td>0.22</td>
</tr>
<tr>
<td>10</td>
<td>0.0219</td>
<td>0.22</td>
</tr>
<tr>
<td>100</td>
<td>0.229</td>
<td>0.23</td>
</tr>
<tr>
<td>$10^3$</td>
<td>3.33</td>
<td>0.33</td>
</tr>
<tr>
<td>$10^4$</td>
<td>136.</td>
<td>1.4</td>
</tr>
<tr>
<td>$10^5$</td>
<td>$1.06 \times 10^4$</td>
<td>10.6</td>
</tr>
<tr>
<td>$10^6$</td>
<td>$5.38 \times 10^5$</td>
<td>53.8</td>
</tr>
<tr>
<td>$10^7$</td>
<td>$9.21 \times 10^6$</td>
<td>92.1</td>
</tr>
</tbody>
</table>
Fig. 5.3  Single-collision energy-loss spectra for 50-eV and 150-eV electrons and 1-MeV protons in liquid water. (Courtesy Oak Ridge National Laboratory, operated by Martin Marietta Energy Systems, Inc., for the Department of Energy.)
Linear Stopping Power – A Semiclassic Treatment

Consider the following diagram

Step 1: Deriving the energy transfer from the heavy charged particle to a free electron nearby.

and assuming the electron is stationary during the collision...

Fig. 5.4 Representation of the sudden collision of a heavy charged particle with an electron, located at the origin of XY coordinate axes shown. See text.
Linear Stopping Power – A Semiclassic Treatment

Step 2: Integrate the energy transfer to all the electrons surrounding the path of the heavy charged particle.

In traversing a distance \(dx\) in a medium having a uniform density of \(n\) electrons per unit volume, the heavy particle encounters \(2\pi nb db dx\) electrons at impact parameters between \(b\) and \(b + db\), as indicated in Fig. 5.5. The energy lost to these electrons per unit distance traveled is therefore \(2\pi nQb db\). The total linear rate of energy-loss is given by

\[
-\frac{dE}{dx} = 2\pi n \int_{Q_{\text{min}}}^{Q_{\text{max}}} Qb \, db = \frac{4\pi k_0^2 z^2 e^4 n}{mV^2} \int_{b_{\text{min}}}^{b_{\text{max}}} \frac{db}{b} = \frac{4\pi k_0^2 z^2 e^4 n}{mV^2} \ln \frac{b_{\text{max}}}{b_{\text{min}}}. \tag{5.15}
\]
Linear Stopping Power of a Medium for Heavy Charged Particles (revisited)

The linear stopping power of a medium is given by the Bethe formula,

\[
\frac{-\,dE}{dx} = \frac{4\pi k_0^2 z^2 e^4 n}{mc^2 \beta^2} \left[ \ln \frac{2mc^2 \beta^2}{I(1 - \beta^2)} - \beta^2 \right].
\]

\[
\frac{-\,dE}{dx} = \frac{4\pi k_0^2 z^2 e^4 n}{mV^2} \ln \frac{mV^2}{hf}.
\]

\(k_0 = 8.99 \times 10^9 \text{ N m}^2 \text{ C}^{-2}\) (Appendix C),
\(z = \) atomic number of the heavy particle,
\(e = \) magnitude of the electron charge,
\(n = \) number of electrons per unit volume in the medium,
\(m = \) electron rest mass,
\(c = \) speed of light in vacuum,
\(\beta = V/c = \) speed of the particle relative to \(c\),
\(I = \) mean excitation energy of the medium.
Mean Excitation Energies

The main excitation energy \( (I) \) for an element having atomic number \( Z \), can be approximately given by

\[
I \approx \begin{cases} 
19.0 \text{ eV}, & Z = 1 \text{ (hydrogen)} \\
11.2 + 11.7 \ Z \text{ eV}, & 2 \leq Z \leq 13 \\
52.8 + 8.71 \ Z \text{ eV}, & Z > 13.
\end{cases}
\]

For compound or mixture,

If there are \( N_i \) atoms cm\(^{-3} \) of an element with atomic number \( Z_i \) and mean excitation energy \( I_i \), then in formula (5.23) one makes the replacement

\[
\ln I = \sum_i N_i Z_i \ln I_i,
\]
Limitation of the Bethe Formula

Since almost all analytical descriptions of the behavior of heavy charged particles are based on the Bethe formula, it is important to realize the limitation of this formula.

\[-\frac{dE}{dx} = \frac{4\pi k_0^2 z^2 e^4 n}{mc^2 \beta^2} \left[ \ln \frac{2mc^2 \beta^2}{I(1-\beta^2)} - \beta^2 \right]\]

◊ Bethe formula is **valid for high energies** as long as the inequality \(\gamma m/M << 1\) holds.

◊ At **low energy**, it fails because the term \(\ln[2mc^2\beta^2/1(1-\beta^2)]-\beta^2\) **eventually becomes negative** giving a negative value for the stopping power.

◊ It does not account for the fact that at low energies, a **charged particle may capture electrons as it moves**, this will reduce its net charge and reduce the stopping power of the medium.

◊ The dependence of the Bethe formula on \(z^2\) implies that a pair of particles, with the same amount of mass but opposite charge, have the same stopping power and range. **Departures from this predication** has been measured and theoretically predicted.
Interactions of Photons with Matter

- Photoelectric effect
- Compton scattering
  - Compton scattering formula
  - Differential Compton scattering cross section.
  - Total Compton scattering cross section.
  - Linear attenuation coefficient.
Linear Stopping Power – A Semiclassic Treatment

Consider the following diagram

Step 1: Deriving the energy transfer from the heavy charged particle to a free electron nearby.

and assuming the electron is stationary during the collision...

Fig. 5.4 Representation of the sudden collision of a heavy charged particle with an electron, located at the origin of XY coordinate axes shown. See text.
Linear Stopping Power – A Semiclassic Treatment

Step 2: Integrate the energy transfer to all the electrons surrounding the path of the heavy charged particle.

Fig. 5.5 Annular cylinder of length $dx$ centered about path of heavy charged particle. See text.

In traversing a distance $dx$ in a medium having a uniform density of $n$ electrons per unit volume, the heavy particle encounters $2\pi nb \, db \, dx$ electrons at impact parameters between $b$ and $b + db$, as indicated in Fig. 5.5. The energy lost to these electrons per unit distance traveled is therefore $2\pi n Qb \, db$. The total linear rate of energy-loss is given by

$$-\frac{dE}{dx} = 2\pi n \int_{Q_{\text{min}}}^{Q_{\text{max}}} Qb \, db = \frac{4\pi k_0^2 z^2 e^4 n}{m V^2} \int_{b_{\text{min}}}^{b_{\text{max}}} \frac{db}{b} = \frac{4\pi k_0^2 z^2 e^4 n}{m V^2} \ln \frac{b_{\text{max}}}{b_{\text{min}}}. \quad (5.15)$$
## Classification of Photon Interactions

### Table 1. Classification of elementary photon interactions.

<table>
<thead>
<tr>
<th>Type of interaction</th>
<th>Absorption</th>
<th>Scattering</th>
</tr>
</thead>
<tbody>
<tr>
<td>Interaction with:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Atomic electrons</td>
<td>Photoelectric effect ( \sigma_{pe} ) ( \sim Z^4(L.E.) ) ( \sim Z^5(H.E.) )</td>
<td>Elastic (Coherent) Rayleigh scattering ( \sigma_R \sim Z^2 ) (L.E.) Compton scattering ( \sigma_C \sim Z )</td>
</tr>
<tr>
<td>Nucleus</td>
<td>Photonuclear reactions ( (\gamma,n),(\gamma,p) ), photofission, etc. ( \sigma_{ph.n.} \sim Z ) (( h\nu \geq 10\text{MeV} ))</td>
<td>Elastic nuclear scattering ( (\gamma,\gamma) \sim Z^2 ) ( (\gamma,\gamma') ) Inelastic nuclear scattering ( (\gamma,\gamma') )</td>
</tr>
<tr>
<td>Electric field</td>
<td>Electron-positron pair production in field of nucleus, ( \sigma_{pair} \sim Z^2 ) (( h\nu \geq 1.02\text{MeV} ))</td>
<td></td>
</tr>
</tbody>
</table>
Photoelectric Effect – Absorption Edges

- Requires **sufficient photon energy** for P.E. interaction.
- Interaction probability decreases dramatically with increasing energy.
- P.E. interaction is significant only for low energy photons, when the photon energy is close to the binding energies of the target atoms.

Figure 2: Total and partial atomic photoeffect of Ag.
Photoelectric Effect Cross Section

Probability of photoelectric absorption per atom is

\[ \sigma \propto \begin{cases} 
\frac{Z^4}{(hv)^{3.5}} & \text{low energy} \\
\frac{Z^5}{(hv)^{3.5}} & \text{high energy}
\end{cases} \]

- The interaction cross section for photoelectric effect depends strongly on $Z$.

- Photoelectric effect is favored at lower photon energies. It is the major interaction process for photons at low hundred keV energy range.
Relaxation Process after Photoelectric Effect

The excited atoms will **de-excite** through one of the following processes:

- Auger electron emission dominates in **low-Z** elements. **Characteristic X-ray** emission dominates in **higher-Z** elements.
Auger Electrons

The relative probability of the emission of characteristic radiation to the emission of an Auger electron is called the fluorescent yield, $\omega$:

$$\omega_K = \frac{\text{Number K x ray photons emitted}}{\text{Number K shell vacancies}}$$  \hspace{1cm} (3-12)

Values for $\omega_K$ are given in Table 3-1. We see that for large $Z$ values fluorescent radiation is favored, while for low values of $Z$ Auger electrons tend to be produced.

From this table we see that if a nucleus with $Z = 40$ had a K shell hole, then on the average 0.74 fluorescent photons and 0.26 Auger electrons would be emitted.

<table>
<thead>
<tr>
<th>TABLE 3-1</th>
<th>Fluorescent Yield</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z$</td>
<td>$\omega_K$</td>
</tr>
<tr>
<td>10</td>
<td>0</td>
</tr>
<tr>
<td>15</td>
<td>.05</td>
</tr>
<tr>
<td>20</td>
<td>.19</td>
</tr>
<tr>
<td>25</td>
<td>.30</td>
</tr>
<tr>
<td>30</td>
<td>.50</td>
</tr>
<tr>
<td>35</td>
<td>.63</td>
</tr>
</tbody>
</table>

From Evans (E1)
Basic Kinematics in Compton Scattering

The **energy transfer** in Compton scattering may be derived as the following:

- Assuming that the **electron binding energy** is small compared with the energy of the incident photon – **elastic scattering**.
- Write out the **conservation of energy and momentum**:

\[
\begin{align*}
\text{Conservation of energy} & \quad h\nu + mc^2 = h\nu' + E' \\
\text{Conservation of momentum} & \quad \frac{h\nu}{c} = \frac{h\nu'}{c} \cos \theta + P' \cos \varphi \\
& \quad \frac{h\nu'}{c} \sin \theta = P' \sin \varphi
\end{align*}
\]
Energy Transfer in Compton Scattering

If we assume that the electron is free and at rest, the scattered gamma ray has an energy

$$hν' = \frac{hν}{1 + \frac{hν}{m_0c^2}(1 - \cos θ)},$$

and the photon transfers part of its energy to the electron (assumed to be at rest before the collision), which is known as the **recoil electron**. Its energy is simply

$$E_{\text{recoil}} = hν - hν' = hν - \frac{hν}{1 + \frac{hν}{m_0c^2}(1 - \cos(θ))}$$

assuming the binding energy of the electron is negligible.

**In the simplified elastic scattering case, there is an one-to-one relationship between scattering angle and energy loss!!**


NPRE 441, Principles of Radiation Protection, Spring 2021
Angular Distribution of the Scattered Gamma Rays

The differential scattering cross section per electron — the probability of a photon scattered into a unit solid angle around the a given scattering angle $\theta$, when the incident photon is passing normally through a thin layer of scattering material that contains one electron per unit area.

\[
\frac{d\sigma}{d\Omega}(\theta) = r_e^2 \left( \frac{1}{1 + \alpha(1 - \cos \theta)} \right)^2 \left( \frac{1 + \cos^2 \theta}{2} \right) \left( 1 + \frac{\alpha^2(1 - \cos \theta)^2}{(1 + \cos^2 \theta)[1 + \alpha(1 - \cos \theta)]} \right) \left( m^2 sr^{-1} \right)
\]

where $\alpha = \frac{hv}{m_0 c^2}$ and $r_e$ is the classical electron radius.

Fig. 5.15. Compton scattering diagram to illustrate differential scattering cross section. $S$ is a sphere of unit radius whose center is the scattering electron.
Total Compton Collision Cross Section for an Electron

Compton Collision Cross Section is defined as the total cross section per electron for Compton scattering. It can be derived by integrating the differential cross section, \( \frac{d\sigma}{d\Omega} \), over \( 4\pi \) solid angle.

Since

\[ d\Omega = 2\pi \sin \theta \, d\theta, \]

then the Compton scattering cross section per electron is given by

\[ \sigma = 2\pi \int_{\Omega} \frac{d\sigma}{d\Omega} \cdot \sin \theta \cdot d\theta \quad (m^2) \]

Note that the Compton scattering cross section per electron is given in unit of \( m^2 \).
Energy Distribution of Compton Recoil Electrons

Given the Klein-Nishina formula, how do we derive the **energy spectrum of recoil electrons**? In other words, how do we derive the probability of a gamma-ray undergoing a Compton scattering and transferring an energy falling into an energy window, $E_{recoil} \in \left[ E' - \frac{1}{2} \Delta E, E' + \frac{1}{2} \Delta E \right]$?

![Compton scattering diagram](image)

**Fig. 5.15.** Compton scattering diagram to illustrate differential scattering cross section. $S$ is a sphere of unit radius whose center is the scattering electron.

$$
\frac{d\sigma}{d\Omega}(\theta) = r_e^2 \left( \frac{1}{1 + \alpha(1 - \cos \theta)} \right) \left( \frac{1 + \cos^2 \theta}{2} \right) \left( 1 + \frac{\alpha^2 (1 - \cos \theta)^2}{(1 + \cos^2 \theta)[1 + \alpha(1 - \cos \theta)]} \right) \left( m^2 sr^{-1} \right)
$$
**Energy Distribution of Compton Recoil Electrons**

Klein-Nishina formula can be used to derive the **energy spectrum of recoil electrons** as the following:

\[
\frac{d\sigma}{dE_{\text{recoil}}} = \frac{d\sigma}{d\Omega} \cdot \frac{d\Omega}{d\theta} \cdot \frac{d\theta}{dE_{\text{recoil}}} \left( \text{m}^2 \cdot \text{keV}^{-1} \right)
\]

If a gamma-ray underwent a Compton Scattering, then probability of the gamma-ray transferring a given amount of energy falling into a small energy window, \(E_{\text{recoil}} \in [E' - \frac{1}{2} \Delta E, E' + \frac{1}{2} \Delta E]\) would be proportional to

\[
p \propto \Delta E \cdot \left. \frac{d\sigma}{dE_{\text{recoil}}} \right|_{E'}
\]
Energy Distribution of Compton Recoil Electrons

The energy distribution for the recoil electrons could be derived with the following differential cross section

\[
\frac{d\sigma}{dE_{\text{recoil}}} = \frac{\sigma}{d\Omega} \frac{d\Omega}{d\theta} \frac{d\theta}{dE_{\text{recoil}}}
\]

The three partial derivative terms on the right-hand side of the equation can be derived from the following relationships:

- **From Klein-Nishina formula:**

  \[
  \frac{d\sigma}{d\Omega}(\theta) = r_e^2 \left( \frac{1}{1+\alpha(1-\cos \theta)} \right)^2 \left( \frac{1+\cos^2 \theta}{2} \right) \left( 1 + \frac{\alpha^2(1-\cos \theta)^2}{(1+\cos^2 \theta)[1+\alpha(1-\cos \theta)]} \right)
  \]

- **From Compton equation:**

  \[
  E_{\text{recoil}} = h\nu - h\nu' = h\nu - \frac{h\nu}{1 + \frac{h\nu}{m_0c^2(1-\cos \theta)}} \quad \Rightarrow
  \]

  \[
  \frac{d\theta}{dE_{\text{recoil}}} = - \frac{m_0c^2}{(hv-E_{\text{recoil}})^2 \cdot \sin \theta} = \frac{m_0c^2}{(hv)^2 \cdot \sin \theta} \left[ 1 + \frac{h\nu}{m_0c^2(1-\cos \theta)} \right]^2
  \]

- **From the known scattering geometry:**

  \[
  d\Omega = 2\pi \sin \theta \, d\theta \quad \Rightarrow \quad \frac{d\Omega}{d\theta} = 2\pi \sin \theta
  \]
Energy Distribution of Compton Recoil Electrons

Therefore, the differential cross section becomes

\[
\frac{d\sigma}{dE_{\text{recoil}}} = \frac{d\sigma}{d\Omega} \frac{d\theta}{dE_{\text{recoil}}}
\]

\[
= \left[ r_e^2 \left( \frac{1}{1+\alpha(1-\cos \theta)} \right)^2 \left( \frac{1+\cos^2 \theta}{2} \right) \left( 1 + \frac{\alpha^2(1-\cos \theta)^2}{(1+\cos^2 \theta)[1+\alpha(1-\cos \theta)]} \right) \right] \times
\]

\[
\left[ \frac{m_0c^2}{(hv)^2 \cdot \sin \theta} \left[ 1 + \frac{hv}{m_0c^2(1-\cos \theta)} \right]^2 \right] \times
\]

\[
[2\pi \sin \theta]
\]

Remember than

\[
E_{\text{recoil}} = hv - hv' = hv - \frac{hv}{1 + \frac{hv}{m_0c^2(1-\cos \theta)}}.
\]

Then \( \frac{d\sigma}{dE_{\text{recoil}}} \) could be written as an explicit function of \( E_{\text{recoil}} \).

\[
\frac{d\sigma}{dE_{\text{recoil}}} (\theta) \Rightarrow \frac{d\sigma}{dE_{\text{recoil}}} (E_{\text{recoil}})
\]
Energy Distribution of Compton Recoil Electrons

Klein-Nishina formula can be used to calculate the energy spectrum of recoil electrons as the following:

$$\frac{d\sigma}{dE_{recoil}} = \frac{d\sigma}{d\Omega} \cdot \frac{d\Omega}{d\theta} \cdot \frac{d\theta}{dE_{recoil}} (m^2 \cdot keV^{-1})$$

If a gamma-ray underwent a Compton Scattering, then probability of the gamma-ray transferring a given amount of energy that fall into a small energy window, $E_{recoil} \in \left[ E' - \frac{1}{2} \Delta E, E' + \frac{1}{2} \Delta E \right]$ would be proportional to

$$P \propto \Delta E \cdot \left( \frac{d\sigma}{dE_{recoil}} \right) \bigg|_{E_{recoil}=E'}$$
Energy Distribution of Compton Recoil Electrons

Remember that the maximum amount of energy that a photon can transfer to an electron in a single Compton scattering is given by:

\[ E_{\text{max}} = \frac{2h\nu}{2 + m_c^2/h\nu} \]

The energy distribution of the recoil electrons derived using the Klein-Nishina formula is closely related to the energy spectrum measured with “small” detectors (in particular, the so-called Compton continuum).

Figure 10.1 Shape of the Compton continuum for various gamma-ray energies. (From S. M. Shafroth (ed.), Scintillation Spectroscopy of Gamma Radiation. Copyright 1964 by Gordon & Breach, Inc. By permission of the publisher.)
Average fraction of energy transfer to the recoil electron through a single Compton Collision

**Average recoil electron energy** $E_{avg\_recoil}$ is of special interest for dosimetry is the, since it is an approximation of the **radiation dose delivered by each photon through a single Compton scattering interaction**.

The **average fraction of energy transfer to the recoil electron** through a single Compton scattering is given by

\[
\frac{E_{avg\_recoil}}{h\nu} = \int_{E_{recoil}}^{E_{recoil}} \frac{E_{recoil}}{h\nu} \cdot \left[\left(\frac{d\sigma}{dE_{recoil}}\right)/\sigma\right] \cdot dE_{recoil},
\]

where $\sigma$ is the Compton scattering cross section per electron and is given by

\[
\sigma = 2\pi \int_{\Omega} \frac{d\sigma}{d\Omega} \cdot \sin \theta \cdot d\theta \ (m^2).
\]
Average fraction of energy transfer to the recoil electron through a single Compton Collision

<table>
<thead>
<tr>
<th>Photon Energy $hν$ (MeV)</th>
<th>Average Recoil Electron Energy $T_{avg}$ (MeV)</th>
<th>Average Fraction of Incident Energy $T_{avg}/hν$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>0.0002</td>
<td>0.0187</td>
</tr>
<tr>
<td>0.02</td>
<td>0.0007</td>
<td>0.0361</td>
</tr>
<tr>
<td>0.04</td>
<td>0.0027</td>
<td>0.0667</td>
</tr>
<tr>
<td>0.06</td>
<td>0.0056</td>
<td>0.0938</td>
</tr>
<tr>
<td>0.08</td>
<td>0.0094</td>
<td>0.1117</td>
</tr>
<tr>
<td>0.10</td>
<td>0.0138</td>
<td>0.138</td>
</tr>
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<td>0.20</td>
<td>0.0432</td>
<td>0.216</td>
</tr>
<tr>
<td>0.40</td>
<td>0.124</td>
<td>0.310</td>
</tr>
<tr>
<td>0.60</td>
<td>0.221</td>
<td>0.368</td>
</tr>
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<td>0.80</td>
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<td>1.00</td>
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<td>0.440</td>
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<td>0.607</td>
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<td>30.4</td>
<td>0.760</td>
</tr>
<tr>
<td>60.0</td>
<td>46.6</td>
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<td>80.0</td>
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<td>0.787</td>
</tr>
<tr>
<td>100.0</td>
<td>79.4</td>
<td>0.794</td>
</tr>
</tbody>
</table>
Total Compton Collision Cross Section for an Electron

**Compton Collision Cross Section** is defined as the total cross section per electron for Compton scattering. It can be derived by integrating the differential cross section, $\frac{d\sigma}{d\Omega}$, over $4\pi$ solid angle.

Since

$$d\Omega = 2\pi \sin \theta \, d\theta,$$

the total Compton scattering cross section per electron is given by

$$\sigma = 2\pi \int_{\Omega} \frac{d\sigma}{d\Omega} \cdot \sin \theta \cdot d\theta \ (m^2)$$

Note that the Compton scattering cross section per electron is given in unit of $m^2$. 
Pair Production

**Definition:**
Pair production refers to the creation of an electron-positron pair by an incident gamma ray in the vicinity of a nucleus.

**Characteristics**
- The minimum energy required is
  \[ E_\gamma \geq 2m_e c^2 + \frac{2m_e^2 c^2}{m_{\text{nucleus}}} \approx 2m_e c^2 = 1.022 \text{MeV} \]
- The process is more probable with a heavy nucleus and incident gamma rays with higher energies.
- The positrons emitted will soon annihilate with ordinary electrons near by and produces two 511keV gamma rays.
Photonuclear Reaction

A photon can be absorbed by an atomic nucleus and knock out a nucleon. This process is called photonuclear reaction. For example,

\[ ^9_4 Be + h\nu \rightarrow ^8_4 Be + ^1_0 n, \quad Q - \text{value} : -1.666\text{MeV} \]
\[ ^2_1 H + h\nu \rightarrow ^1_1 H + ^1_0 n, \quad Q - \text{value} : -2.226\text{MeV} \]

The photon must possess enough energy to overcome the nuclear binding energy, which is generally several MeV.

The threshold, or the minimum photon energy required, for \((\gamma,p)\) reaction is generally higher than that for \((\gamma,n)\) reactions. Since the repulsive Coulomb barrier that a proton must overcome to escape from the nucleus.

Other nuclear reactions are also possible, such as \((\gamma, 2n)\), \((\gamma, np)\), \((\gamma, \alpha)\) and photon induced fission reaction.
Interaction of Photons in Matter

Figure 2.18 Energy dependence of the various gamma-ray interaction processes in sodium iodide. (From The Atomic Nucleus by R. D. Evans Copyright 1955 by the McGraw-Hill Book Company. Used with permission.)
Energy Absorption Coefficient

Linear attenuation coefficient

Energy absorption coefficient

  What is energy transfer coefficient?

  What is energy absorption coefficient?

  Application of energy absorption coefficient in dose calculation
Linear Attenuation Coefficients

Linear attenuation coefficient is measured using the following setup

\[ I = I_0 e^{-\mu x} \]

**FIGURE 8.7.** Illustration of “good” scattering geometry for measuring linear attenuation coefficient \( \mu \). Photons from a narrow beam that are absorbed or scattered by the absorber do not reach a small detector placed in beam line some distance away.

Using small detector to avoid the effect of Compton scattered photons on the measured linear attenuation coefficient.
Linear Attenuation Coefficients

- **Linear attenuation coefficient** for a given material comprises the individual contributions from various physical processes,

\[ \mu = \tau_{\text{photoelectric}} + \sigma_{\text{Compton}} + K_{\text{pair}} \]

- **Mass attenuation coefficient** is simply defined as,

\[ \mu_m = \frac{\mu}{\rho} \text{ (cm}^2 / \text{g)} \]

As shown in the discussion of the attenuation of beta and alpha particles, the mass attenuation coefficient is used to partially remove the dependence on different atomic compositions and densities, and provides in an unified measure of photon attenuation amongst various materials.
Energy Transfer by a Gamma Ray Beam

Compton scattering

- All Compton scattered gamma rays escaped
- Multiple Compton scattering ignored

Photoelectric effect

- All characteristic X-rays escaped
- All photoelectrons, auger electrons and Compton recoil electrons are absorbed

Pair production

- All annihilation gamma rays escaped
Energy-Transfer Coefficient

The total mass energy transfer coefficient is given by

\[
\frac{\mu_{tr}}{\rho} = \frac{\tau}{\rho} \left( 1 - \frac{\delta}{h\nu} \right) + \frac{\sigma}{\rho} \left( \frac{E_{avg}}{h\nu} \right) + \frac{\kappa}{\rho} \left( 1 - \frac{2mc^2}{h\nu} \right)
\]

The fraction of energy that is carried away by characteristic x-rays following photoelectric effect.

The fraction of energy that is carried away by the two 511keV gamma rays generated by the annihilation of the positron.

The fraction of energy that is transferred to recoil electron through Compton scattering.

For a parallel beam of monochromatic gamma rays transmitting through a unit distance in an absorbing material, the energy-transfer coefficient is the fraction of energy that was originally carried by the incident gamma ray beam and transferred into the kinetic energy of secondary electron inside the absorber.
Comparison Between Linear Attenuation Coefficient and Energy Absorption Coefficient

**Figure 8.13.** Linear attenuation and energy-absorption coefficients as functions of energy for photons in water.

The fraction of energy carried by the photons being absorbed in material.

The fraction of photons removed from the beam after traveling through a unit distance.
Energy-Transfer and Energy-Absorption Coefficients

The photon fluence $\Phi$: the number of photons cross a unit area perpendicular to the beam.

The photon fluence rate: the number of photons per unit area per unit time.

$$\Phi = \frac{d\Phi}{dt} \left( m^{-2} s^{-1} \right)$$

The energy fluence $\Psi \ (J m^{-2})$: the amount of energy passes per unit area perpendicular to the beam.

The energy fluence rate ($J m^{-2} s^{-1}$): the amount of energy transfer per unit area per unit time.

$$\dot{\Psi} = \frac{d\Psi}{dt} \left( J m^{-2} s^{-1} \right)$$
Calculation of Energy Transfer and Energy Absorption

For simplicity, we consider an idealized case, in which

- Photons are assumed to be monoenergetic and in broad parallel beam.
- Multiple Compton scattering of photons is negligible.
- Virtually all fluorescence and bremsstrahlung photons escape from the absorber.
- All secondary electrons (Auger electrons, photoelectrons and Compton electrons) generated are stopped in the slab.

Under these conditions, the transmitted energy intensity (the amount of energy transmitted through a unit area within each second) can be given by

\[ \dot{\Psi} = \dot{\Psi}_0 e^{-\mu_{en}x} \]
Calculation of Energy Transfer and Energy Absorption

Assuming $\mu_{en}x << 1$, which is consistent with the thin slab approximation and the energy fluence rate carried by the incident gamma ray beam is $\dot{\Psi}_0 (J \cdot cm^{-2} \cdot s^{-1})$. Then the energy absorbed in the thin slab per second over a unit cross section area is given by

$$\dot{\Psi}_0 \mu_{en} x (J \cdot cm^{-2} \cdot s^{-1})$$

The rate of energy absorbed in the slab of area $A (cm^2)$ and thickness $x$ is

$$A \dot{\Psi}_0 \mu_{en} x (J \cdot s^{-1})$$

Given the density of the material is $\rho$, the rate of energy absorption per unit mass (Dose Rate) in the slab is

$$\dot{D} = \frac{A(cm^2) \cdot \dot{\Psi}_0 (J \cdot cm^{-2} \cdot s^{-1}) \cdot \mu_{en}(cm^{-1}) \cdot x (cm)}{\rho (g \cdot cm^{-3}) \cdot A(cm^2) \cdot x(cm)}$$

*Dose rate in the absorber:* $\dot{D} = \dot{\Psi}_0 \frac{\mu_{en}}{\rho} (J \cdot g^{-3} \cdot s^{-1})$
Interactions of Neutrons with Matter

Reading Material:

- Chapter 5 in "Introduction to Health Physics", Third edition, by Cember.
Chapter 2: Interaction of Radiation with Matter – Interaction of Neutrons with Matter

Interaction of Neutrons – Learning Objectives

- Elastic scattering of neutrons
  - Energy transfer as a function of scattering angle.
  - Angular distribution of scattered neutrons.
  - Energy spectrum of scattered neutrons and average energy loss of a neutron through a single elastic scattering.
- Neutron induced nuclear reaction
  - Several important neutron induced nuclear reactions.
  - Endothermic reactions and threshold energy.
  - Neutron activation and the derivation of the neutron induced activity.
Elastic Scattering of Neutrons

The elastic scattering plays an important role in neutron energy measurements. For example, a proton-neutron telescope illustrated below can be used to accurately measure the spectrum of neutrons in a collimated beam.

\[
E_{\text{proton}} = E_{\text{neutron}} \cos^2 \theta
\]

**FIGURE 10.36.** Arrangement of proton-recoil telescope for measuring spectrum neutron beam.
Elastic Scattering of Neutrons

The **maximum energy** that a neutron of mass $M$ and kinetic energy $E_n$ can transfer to a nucleus of mass $m$ in a single elastic collision given by

$$E_{\text{max}} = E_n \frac{4Mm}{(M + m)^2}$$

<table>
<thead>
<tr>
<th>Nucleus</th>
<th>$Q_{\text{max}}/E_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^1\text{H}$</td>
<td>1.000</td>
</tr>
<tr>
<td>$^2\text{H}$</td>
<td>0.889</td>
</tr>
<tr>
<td>$^4\text{He}$</td>
<td>0.640</td>
</tr>
<tr>
<td>$^9\text{Be}$</td>
<td>0.360</td>
</tr>
<tr>
<td>$^{12}\text{C}$</td>
<td>0.284</td>
</tr>
<tr>
<td>$^{16}\text{O}$</td>
<td>0.221</td>
</tr>
<tr>
<td>$^{56}\text{Fe}$</td>
<td>0.069</td>
</tr>
<tr>
<td>$^{118}\text{Sn}$</td>
<td>0.033</td>
</tr>
<tr>
<td>$^{238}\text{U}$</td>
<td>0.017</td>
</tr>
</tbody>
</table>
Energy Transfer in Compton Scattering (Revisited)

If we assume that the electron is free and at rest, the scattered gamma ray has an energy

\[ h\nu' = \frac{h\nu}{1 + \frac{h\nu}{m_0c^2}(1 - \cos \theta)}, \]

and the photon transfers part of its energy to the electron (assumed to be at rest before the collision), which is known as the *recoil electron*. Its energy is simply

\[ E_{\text{recoil}} = h\nu - h\nu' = h\nu - \frac{h\nu}{1 + \frac{h\nu}{m_0c^2}(1 - \cos(\theta))} \]

assuming the binding energy of the electron is negligible.

In the simplified elastic scattering case, there is an one-to-one relationship between scattering angle and energy loss!!


NPRE 441, Principles of Radiation Protection, Spring 2021
Angular Distribution of the Scattered Gamma Rays (Revisited)

The differential scattering cross section – the probability of a photon scattered into a unit solid angle around the scattering angle $\theta$, when passing normally through a layer of material containing one electron per unit area.

$$\frac{d\sigma}{d\Omega}(\theta) = r_e^2 \left( \frac{1}{1 + \alpha (1 - \cos \theta)} \right)^2 \left( \frac{1 + \cos^2 \theta}{2} \right) \left( 1 + \frac{\alpha^2 (1 - \cos \theta)^2}{(1 + \cos^2 \theta)[1 + \alpha (1 - \cos \theta)]} \right) (m^2 sr^{-1})$$

Fig. 5.15. Compton scattering diagram to illustrate differential scattering cross section. $S$ is a sphere of unit radius whose center is the scattering electron.
Elastic Scattering of Neutrons

- The **elastic scattering** is the dominating mechanism whereby fast neutrons deliver dose to tissue.

- The recoil nuclei are essentially ionizing particles traveling in media and losing their energy through ionization and excitation.

- As we will see later, over 85% of the “first-collision” dose in soft tissue (composed of H, C, O and N) arises from n-p scattering for neutron energy below 10MeV.
Elastic Scattering of Neutrons

Kinematics of neutron scattering:

- Energy transfer as a function of scattering angle.
- Angular distribution of scattered neutrons.
- Energy spectrum of scattered neutrons.
- Average logarithm energy decrement of a neutron in multiple scattering.
Angular Distributions of the Scattered Neutrons

For neutron energies up to 10MeV, it is experimentally observed that the scattering of neutrons is isotropic in the center-of-mass coordinate system. The neutron and the recoil nuclei are scattered with equal probability in any direction in this 3-D coordinate system.

\[
d\Omega = \frac{2\pi r \sin \theta \ r d\theta}{r^2}
\]
Angular Distributions of the Scattered Neutrons

Neutron

$m, \vec{V}$

M

Center-of-mass

Recoil nucleus

θ

ω

Neutron

θ

NPRE 441, Principles of Radiation Protection, Spring 2021
Angular Distributions of the Scattered Neutrons

Before collision, in laboratory system

\[ \vec{v}_{\text{C}} = \frac{m}{M + m} \vec{v}_0 \]

Before collision, in the center-of-mass (CM) system

\[ \vec{v}_{\text{C}} = \frac{M}{M + m} \vec{v}_0 \]

After collision, in laboratory system

\[ \vec{v}_1 = \frac{M}{M + m} \vec{v}_0 \]

\[ \vec{v}' = \frac{m}{M + m} \vec{v}_0 \]

\[ \theta: \text{scattering angle in the CM system} \]

\[ m, v_0 \]

\[ M, 0 \]

\[ M, 0 \]

\[ m \]

\[ M \]
Energy Spectrum of Scattered Neutrons

Average energy carried by the scattered neutron:

\[ E'_{avg} = \frac{1+\alpha}{2} E_0 = \frac{1+\frac{(M-m)^2}{(M+m)^2}}{2} E_0 = \frac{M^2+m^2}{(M+m)^2} \cdot E_0 \]

\[ \alpha = \frac{(M-m)^2}{(M+m)^2} \]

Average energy transferred to the recoil nucleus:

\[ E_{avg\_energy\_loss} = E_0 - E'_{avg} = \frac{2Mm}{(M + m)^2} \cdot E_0 \]
Fast- and Thermal-Diffusion Lengths

The fast-diffusion length: the average straight line distance covered by fast neutrons traveling in a given medium.

The thermal-diffusion length: the average distance covered by thermalized neutrons before it is absorbed. It is measured by the thickness of a slowing down medium that attenuates the beam of thermal neutrons by a factor of e. Thus the attenuation of a beam of thermal neutrons by a substance of thickness \( t \) (cm), whose thermal diffusion length is \( L \) (cm), is given by

\[
\begin{align*}
n &= n_0 e^{-t/L} 
\end{align*}
\]

<table>
<thead>
<tr>
<th>Substance</th>
<th>Fast Diffusion Length, cm</th>
<th>Thermal Diffusion Length, cm</th>
</tr>
</thead>
<tbody>
<tr>
<td>H_2O</td>
<td>5.75</td>
<td>2.88</td>
</tr>
<tr>
<td>D_2O</td>
<td>11</td>
<td>171</td>
</tr>
<tr>
<td>Be</td>
<td>9.9</td>
<td>24</td>
</tr>
<tr>
<td>C (graphite)</td>
<td>17.3</td>
<td>50</td>
</tr>
</tbody>
</table>
Interaction of Slow Neutrons (E<0.5eV)

- Significant interactions include *elastic scattering* and *neutron induced nuclear reactions*.
- Due to the low neutron energy, very little energy can be transferred by elastic scattering.
- The more significant effect of elastic scattering is to *slow down* slow neutrons and turn them into *thermal neutrons* (average E<0.025eV at room temperature).
- Neutron absorption followed by the immediate emission of a gamma ray photon and other particles.
Interaction of Slow Neutrons (E<0.5eV)

The most important interactions between slow neutrons and absorbing materials are *neutron-induced reactions*, such as \((n,\gamma)\), \((n,\alpha)\), \((n,p)\) and \((n, \text{fission})\) etc. These interactions lead to more prominent signatures for neutron detection.

\[
\text{neutron + target nucleus} \Rightarrow \begin{cases} 
\text{recoil nucleus} \\
\text{protons} \\
\text{alpha particles} \\
\text{fission fragments}
\end{cases}
\]
Neutron Induced Reactions

\[ _0^1n+_1^1H \rightarrow _1^2H+_0^0\gamma \]

Neutron absorption followed by the immediate emission of a gamma ray photon.

Since the thermal neutron has negligible energy by comparison, the gamma photon has the energy \( Q=2.22\text{MeV} \) released by the reaction, which represents the binding energy of the deuteron.

The capture cross section per atom is 0.33barn.

When tissue is exposed to thermal neutrons, this reaction provides a source of gamma rays that delivers dose to the tissue.
Neutron Induced Reactions

\[ _0^1n + _7^{14}N \rightarrow _6^{14}C + _1^1p \]

- Cross section for thermal neutron is 1.70 barns.
- Q=0.626MeV.
- Since the range of the proton and the $^{14}C$ nucleus are relatively small, their energy is deposited locally at the site where the neutron was captured.
- Capture by hydrogen and nitrogen are the only two processes through which neutron deliver a significant does to soft tissue.
Neutron Induced Reactions

\[ \frac{1}{0} n + ^{23}_{11}Na \rightarrow ^{24}_{11}Na + \gamma \]

- Cross section for thermal neutron is 0.534 barns.
- Q=0.626MeV.
- \(^{24}\)Na undergo radioactive decay with the emission of two gamma rays, having energies of 2.75MeV and 1.37MeV per disintegration.
- Since \(^{23}\)Na is a normal constituent of blood, activation of blood sodium can be sued as a dosimetric tool when persons are exposed to relatively high does of neutrons, for example, in a criticality accident.
Energetics of Threshold Reactions

Consider the following reaction

\[ ^0_1n + ^{32}_{16}S \rightarrow ^{32}_{15}P + ^1_1p \]

The neutrons must have an energy of above a certain threshold to enable this reaction.

These reactions are called endothermic reactions, in which energy is converted into mass and therefore Q<0.
Threshold Reactions

Energy release: \( Q = M_1 + M_2 - (M_3 + M_4) \) (1)

Conservation of energy: \( E_1 = E_3 + E_4 + Q \Rightarrow E_4 = E_1 - E_3 \) (2)

Conservation of momentum: \( p_1 = p_3 + p_4 \Rightarrow (2M_1 E_1)^{1/2} = (2M_3 E_3)^{1/2} + (2M_4 E_4)^{1/2} \) (3)

Substitute (2) into (3), we have

\[ (2M_1 E_1)^{1/2} = (2M_3 E_3)^{1/2} + (2M_4 (E_1 - E_3))^{1/2}. \]

After some algebraic manipulations,

\[ E_3 - \frac{2(M_1 M_3 E_1)^{1/2}}{M_3 + M_4} \sqrt{E_3} - \frac{(M_4 - M_1) E_1 + M_4 Q}{M_3 + M_4} = 0. \]

For \( E_3 \) to take a real value, we need to have

\[ \left[-\frac{2(M_1 M_3 E_1)^{1/2}}{M_3 + M_4} \right]^2 - 4 \left[-\frac{(M_4 - M_1) E_1 + M_4 Q}{M_3 + M_4} \right] \geq 0, \]

or

\[ \frac{M_1 M_3 E_1}{(M_3 + M_4)^2} + \frac{(M_4 - M_1) E_1 + M_4 Q}{M_3 + M_4} \geq 0, \]

which finally leads to

\[ E_i \geq -Q \left(1 + \frac{M_1}{M_3 + M_4 - M_1}\right). \]
Neutron Activation

Considering the decay of the radioactive daughters and the constant bombardment by incident neutrons, the net rate of increase of radioactive atoms is given by

\[ \frac{dN}{dt} = \phi \sigma n - \lambda N, \]

where \( \phi \) = flux, neutrons per cm\(^2\) per s,
\( \sigma \) = activation cross section, cm\(^2\),
\( \lambda \) = transformation constant of the induced activity,
\( N \) = number of radioactive atoms,
\( n \) = number of target atoms.

The radioactivity induced by neutron activation (the number of disintegration of the activated daughter atoms per second) is given by

\[ \lambda N = \phi \sigma n (1 - e^{-\lambda t}) \]
Neutron Activation

The radioactivity induced by neutron activation (the number of disintegration of the activated daughter atoms per second) is given by

$$\lambda N = \phi \sigma n (1 - e^{-\lambda t})$$

The saturation activity is given by $\phi \sigma n$. For an infinitely long irradiation time, it represents the maximum obtainable activity with any given neutron flux.

The analysis leading to these results is identical to that used for analyzing the secular equilibrium for radioactive decay chains, in which the daughter has a much shorter decay time than that of the parent.
**Neutron Activation**

**FIGURE 9.8.** Buildup of induced activity $\lambda N$, as given by Eq. (9.36), during neutron irradiation at constant fluence rate.
Chapter 3: Counting Statistics
Statistics of Radiation and Radiation Detection

Statistical nature of radiation and radiation interaction:

❖ How much energy will a 1 MeV photon lose in its next collision with an atomic electron?
❖ Will a 400keV photon penetrate a 2 mm lead shielding without interaction?
❖ When we use measured count-rate to estimate the activity of a source, and how certain are we on the estimation?
Exponential Radioactive Decay

Sample activity (A)

- True sample activity is never known.
- The best we can do is to repeat the counting process for a number of times and use the average as an indication of the sample activity – average number of decays in the sample per second.

\[ A = A_0 e^{-\lambda t} \]

- The above equation can be interpreted by implying that the probability that an atom survives a time \( t \) without disintegration is
  \[ q = \text{probability of survival} = e^{-\lambda t} \]
  and
  \[ p = \text{probability of decay} = 1 - q = 1 - e^{-\lambda t} \]

The actual number of decay events is fluctuating around the average value predicted by this equation.
Radioactive Disintegration – Bernoulli Process

Consider the radioactive disintegration process in a sample, it follows the following four conditions:

- It consists of N trials.
- Each trial has a binary outcome: success or failure (decay or not).
- The probability of success (decay) is a constant from trial to trial – all atoms have equal probability to decay.
- The trials are independent.

In statistics, these four conditions characterize a Bernoulli process.
Binomial Distribution

Given, p, N and t, what is the probability of observing n disintegrations within a time t?

The number of ways to chose n atoms from a total of N atoms in the sample is

\[
\binom{N}{n} = \frac{N!}{n!(N-n)!}
\]

So the probability of the n atoms chosen decayed during the time span t is

\[
P_n = \binom{N}{n} p^n q^{N-n}
\]

The above equation describes the so-called Binomial distribution.

What are the mean and standard deviation of a Binomial distribution?
Binomial Distribution

Mean

The mean value $\mu$ of the binomial distribution is defined by Eq. (11.15):

$$
\mu \equiv \sum_{n=0}^{N} nP_n = \sum_{n=0}^{N} n \binom{N}{n} p^n q^{N-n}.
$$

To evaluate this sum, we first use the binomial expansion to write, for an arbitrary (continuous) variable $x$,

$$
(px + q)^N = \sum_{n=0}^{N} \binom{N}{n} p^n x^n q^{N-n} = \sum_{n=0}^{N} x^n P_n.
$$

Differentiation with respect to $x$ gives

$$
Np(px + q)^{N-1} = \sum_{n=0}^{N} nx^{n-1} P_n = \sum_{n=0}^{N} nP_n
$$

Letting $x = 1$ and remembering that $p + q = 1$ gives

$$
Np = \sum_{n=0}^{N} nP_n \equiv \mu.
$$
Binomial Distribution

For a binomial distribution, the mean or the expectation of the number of disintegration in time $t$ is given by

$$\mu \equiv \sum_{n=0}^{N} n \cdot P_n = \sum_{n=0}^{N} n \cdot \binom{N}{n} p^n q^{N-n} = Np$$

and the fluctuation on the number of disintegrations is given by the variance or the standard deviation of the

$$\sigma^2 \equiv \sum_{n=0}^{N} (n - \mu)^2 \cdot P_n = Npq$$

and

$$\sigma \equiv \sqrt{\sum_{n=0}^{N} (n - \mu)^2 \cdot P_n} = \sqrt{Npq}$$
An Example Binomial Distribution

Example

More realistically, consider a $^{42}$K source with an activity of 37 Bq (= 1 nCi). The source is placed in a counter, having an efficiency of 100%, and the numbers of counts in one-second intervals are registered.

(a) What is the mean disintegration rate?
(b) Calculate the standard deviation of the disintegration rate.
(c) What is the probability that exactly 40 counts will be observed in any second?

The decay constant for $^{42}$K is $\lambda = 0.0559 h^{-1} = 1.55 \times 10^{-5} s^{-1}$
Binomial Distribution

For a binomial distribution, the mean or the expectation of the number of disintegration in time $t$ is given by

$$
\mu \equiv \sum_{n=0}^{N} n \cdot P_n = \sum_{n=0}^{N} n \cdot \binom{N}{n} p^n q^{N-n} = Np
$$

and the fluctuation on the number of disintegrations is given by the variance or the standard deviation of the

$$
\sigma^2 \equiv \sum_{n=0}^{N} (n - \mu)^2 \cdot P_n = Npq
$$

$$
\sigma \equiv \sqrt{\sum_{n=0}^{N} (n - \mu)^2 \cdot P_n} = \sqrt{Npq}
$$
Chapter 3: Counting Statistics

An Example Binomial Distribution

Solution
(a) The mean disintegration rate is the given activity, \( r_d = 37 \, s^{-1} \).
(b) The standard deviation of the disintegration rate is given by Eq. (11.18). We work with the time interval, \( t = 1 \, s \). Since the decay constant is \( \lambda = 0.0559 \, h^{-1} = 1.55 \times 10^{-5} \, s^{-1} \), we have

\[
q = e^{-\lambda t} = e^{-1.55 \times 10^{-5} \times 1} = 0.9999845
\]  

and \( p = 1 - q = 0.0000155. \)† The number of atoms present is

\[
N = \frac{r_d}{\lambda} = \frac{37 \, s^{-1}}{1.55 \times 10^{-5} \, s^{-1}} = 2.39 \times 10^6.
\]  

From Eq. (11.18), we obtain for the standard deviation of the disintegration rate

\[
\sigma_{dr} = \frac{\sqrt{Npq}}{t} = \frac{\sqrt{2.39 \times 10^6 \times 0.0000155 \times 0.9999845}}{1 \, s} = 6.09 \, s^{-1},
\]

which is about 16% of the mean disintegration rate.
(c) The probability of observing exactly \( n = 40 \) counts in 1 s is given by Eq. (11.13). However, the factors quickly become unwieldy when \( N \) is not small (e.g., \( 69! = 1.71 \times 10^{98} \)). For large \( N \) and small \( n \), as we have here, we can write for the binomial coefficient

\[
\binom{N}{n} = \frac{N(N - 1) \cdots (N - n + 1)}{n!} \approx \frac{N^n}{n!}, \quad (11.29)
\]

since each of the \( n \) factors in the numerator is negligibly different from \( N \). Equation (11.13) then gives

\[
P_{40} = \frac{(2.39 \times 10^6)^{40}}{40!} (0.0000155)^{40} (0.9999845)^{2.39 \times 10^6 - 40} \quad (11.30)
\]

\[
= \frac{(2.39)^{40} (10^{240}) (0.0000155)^{40} (0.9999845)^{2.39 \times 10^6}}{40!}, \quad (11.31)
\]

where \( n = 40 \ll N \) has been dropped from the last exponent. The right-hand side can be conveniently evaluated with the help of logarithms. To reduce round-off errors, we use four decimal places:

\[
\log (2.39)^{40} = 15.1359 \\
\log (10)^{240} = 240.0000 \\
\log (0.0000155)^{40} = -192.3867 \\
\log (0.9999845)^{2.39 \times 10^6} = -16.0886 \\
-\log 40! = -47.9116 \\
\log P_{40} = -1.251. \quad (11.32)
\]

Thus, \( P_{40} = 10^{-1.251} = 0.0561 \).
Poisson Process

The counting statistics related to nuclear decay processes is often more conveniently described by the Poisson distribution, is related to situations that involves a collection of multiple trials that satisfy the following conditions:

1. The number of trials, N, is very large, e.g. N>>1.
2. Each trial is independent.
3. The probability that each single trial is successful is a constant and approaching zero, p<<1. So the number of successful trials is fluctuating around a finite number.
Binomial Distribution and Poisson Distribution

**Binomial distribution**
The probability of observing $n$ successful trails out of a total of $N$ independent trails:

$$P_n = \binom{N}{n} p^n q^{N-n}$$

mean of the observed number of successful trails :

$$\mu = \sum_{n=0}^{N} n \cdot P_n = \sum_{n=0}^{N} n \cdot \binom{N}{n} p^n q^{N-n} = Np$$

Standard deviation:

$$Std(n) = \sqrt{\sum_{n=0}^{N} (n - \mu)^2 \cdot P_n} = \sqrt{Npq}$$

Gaussian distribution, If $N$ is further increased, and $p$ is further decreased

$$p(x | \mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

**Poisson distribution when**

$N \gg 1$, $p \ll 1$

$$P(n | \mu) = \frac{\mu^n}{n!} e^{-\mu}$$

Mean of $n$:

$$\mu(n) = \mu = N \cdot p$$

Standard deviation :

$$\sigma = \sqrt{\mu} = \sqrt{Np}$$
Chapter 3: Counting Statistics

Poisson Distribution

Remember the conditions for Binomial distribution to be approximated by Poisson Distribution:

1. The number of trials, \( N \), is very large, e.g. \( N >> 1 \).
2. Each trial is independent.
3. The probability that each single trial is successful is a constant and approaching zero, \( p << 1 \). So the number of successful trials is fluctuating around a finite number.
Poisson Distribution

The probability of having $n$ successful trials can be approximated with the Poisson distribution.

$$P(n | \mu) = \frac{\mu^n}{n!} e^{-\mu}$$

and the mean and the variance of number of successful trial are given by

$$Mean(n) = \mu = N \cdot p$$

$$Std(n) \equiv \sigma = \sqrt{\mu} = \sqrt{Np}$$
Binomial Distribution and Poisson Distribution

An example: Consider the following particle counting experiment.

- The detector covers 10% solid angle.
- Detection efficiency: \( \lambda = 55\% \).
- The measurement takes \( T = 1 \) s.
- \( N \) particles reached the detector.
- Detected \( k = 0 \) count.

What do we learn from this experiment?
Binomial Distribution and Poisson Distribution

Suppose there are, in average, \( m \) particles reaching the detector during the given time period \( T \), the probability \( P(N/m) \) of having \( N \) particles reaching the detector during a given experiment would follow ...

\[
P(N|m) = \frac{m^N}{N!} e^{-m}.
\]

Once the \( N \) particles reached the detector, the number of particles detected would follow ...

the Binomial distribution, so that the probability of detecting \( k \) particles is

\[
P(k|N) = \binom{N}{k} \lambda^k (1 - \lambda)^{N-k}.
\]
Poisson Distribution

Therefore, the total probability of detecting $k$ counts is

$$P(k) = \sum_{N=k}^{\infty} P(k|N)P(N|m)$$

$$= \sum_{N=k}^{\infty} \frac{N!}{(N-k)!k!} \lambda^k (1 - \lambda)^{N-k} \cdot \frac{m^N}{N!} e^{-m}$$

$$= \frac{(m\lambda)^k}{k!} e^{-(m\lambda)}$$

If we would like to ensure 90% chance of detecting at least 1 particle, then we could set

$$1 - P(k = 0) = e^{-m\lambda} = 0.9,$$

then the mean number of particles reaching the detector during the measurement should be

$$m = 4.2.$$
So finally, we can conclude that:

Because we did not record any count, we have 90% confidence to claim that the source strength (average number of particles emitted per second) should not exceed

\[
A \leq \frac{4.2}{10\% \cdot 1\ s} = 42 \text{ (particles per sec)} = 42 \text{ Bq}
\]
Semiconductor Detector Configurations

High-purity germanium (HPGe) detectors

- Supper-pure material available, for example 1 part in $10^{12}$
- Depletion depth of $>1\text{cm} \rightarrow$ good detection volume.
- Requires cooling to liquid nitrogen temperature to reduce leakage current.

![Graph showing gamma-ray spectra](image)

**Figure 10.29.** Comparison of gamma-ray spectra from a solution containing radio-nuclides as measured with a NaI scintillator (upper curve) and a Ge(Li) detector. [Reprinted with permission from *A Handbook of Radioactivity Measurements*, NCRP Report No. 58, p. 240, National Council on Radiation Protection and Measurements, Washington, D.C. (1978). Copyright 1978 National Council on Radiation Protection and Measurements.]
Signal Generation by Ionizing Radiation in Semiconductors

The Fano factor.

\[ F \equiv \frac{\text{observed statistical variance}}{E/\epsilon} \]

→ For a given energy deposition \( E \) in the detector and a known energy \( \epsilon \) required to create an e-h pair, the observed fluctuation in the number of charge carriers created is smaller than the one predicted by the Poisson statistics.

→ The measured Fano factors: 0.143 for silicon, 0.129 for germanium, 0.1 for CdZnTe and ~0.1 for HgI₂.

→ In comparison, the Fano factors for scintillators are ~1.
Binomial Distribution and Poisson Distribution

**Binomial distribution**

The probability of observing $n$ successful trials out of a total of $N$ independent trails:

$$P_n = \binom{N}{n} p^n q^{N-n}$$

Mean of the observed number of successful trials:

$$\mu = \sum_{n=0}^{N} n \cdot P_n = \sum_{n=0}^{N} n \cdot \binom{N}{n} p^n q^{N-n} = Np$$

Standard deviation:

$$Std(n) = \sqrt{\sum_{n=0}^{N} (n - \mu)^2 \cdot P_n} = \sqrt{Npq}$$

**Poisson distribution when $N \gg 1$, $p << 1$**

$$P(n | \mu) = \frac{\mu^n}{n!} e^{-\mu}$$

Mean of $n$:

$$\mu(n) = \mu = N \cdot p$$

Standard deviation:

$$\sigma = \sqrt{\mu} = \sqrt{Np}$$
Poisson Distribution

The probability of having $n$ successful trials can be approximated with the Poisson distribution.

$$P(n \mid \mu) = \frac{\mu^n}{n!} e^{-\mu}$$

and the mean and the variance of number of successful trial are given by

$$Mean(n) = \mu = N \cdot p$$

$$Std(n) \equiv \sigma = \sqrt{\mu} = \sqrt{Np}$$
The Gaussian (Normal) Distribution

As \( p \) (the prob. of an atom decay within \( t \)) is getting even smaller and \( N \) is getting larger, both Binomial and Poisson distributions are approaching an extremely useful form of distribution – the Gaussian distribution.

Gaussian distribution is defined for a continuous variable \( x \)

\[
p(x \mid \mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}
\]

but it is very useful for describing the counting fluctuation on discrete numbers.
Chapter 3: Counting Statistics

Poisson Distribution

**FIGURE 11.1.** Comparison of binomial (histogram) and Poisson (solid bars) distributions, having the same mean, $\mu = 10$, but different values of the probability of success $p$ and sample size $N$. The ordinate in each panel shows the probability $P_n$ of exactly $n$ successes, shown on the abscissa. With fixed $\mu$, the Poisson distribution is the same throughout. (Courtesy James S. Bogard.)

**FIGURE 11.2.** Comparison of binomial (histogram) and Poisson (solid bars) distributions for fixed $N$ and different $p$. The ordinate shows $P_n$ and the abscissa, $n$. The mean of the two distributions in a given panel is the same. (Courtesy James S. Bogard.)
Central Limit Theorem

The sum or average of a large number of independent random variables tends to follow Gaussian (Normal) distribution.

The distribution of an average tends to be NORMAL, even when the distribution of the underlying variables from which the average is computed is decidedly non-Normal!
Central Limit Theorem

Consider a series of independent and identically distributed (i.i.d.) random variables, $x_1, x_2, \ldots, x_n$, whose probability density function are given by

$$p_n(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$
An Example of Central Limit Theorem

\[ \bar{X} = \frac{\sum x_n}{n} \]

- NonNormal Distribution of X
- Distribution of Xbar when sample size is 2
- Distribution of Xbar when sample size is 3
- Distribution of Xbar when sample size is 4
- Distribution of Xbar when sample size is 8
- Distribution of Xbar when sample size is 16
- Distribution of Xbar when sample size is 32
Error and Error Propagation

Two ways to express the error associated with a given measurement:

Probable error:

- The symmetric range about the mean, within which there is 50% chance that a measurement will fall.
- The width of the range depends on the distribution of the variable. For example, for Gaussian distributed error, the probable error is $\pm 0.675 \sigma$.

Fractional standard deviation:

- The ratio of the standard deviation and the mean of the distribution of the random variable.
- For Poisson distributed random variable, the fractional standard deviation is simply

\[
\frac{\sigma}{\mu} = \frac{1}{\sqrt{\mu}}
\]
Error Propagation

In some situations, the variable of interest (Q) is not measured directly, but derived as a function of more than one independent random variable whose values are directly measured. The error on the measured values is propagated into the uncertainty on the resultant quantity Q.

Suppose a quantity Q(x,y) that depends on two independent random variables x and y.

The sample mean and variance of variables x and y are derived as $\sigma_x$ and $\sigma_y$ by repeating measurements.

The standard deviation of the indirect quantity Q is approximately given by

$$\sigma_Q^2 \approx \left( \frac{\partial Q}{\partial x} \right)^2 \sigma_x^2 \left( \frac{\partial Q}{\partial y} \right)^2 \sigma_y^2$$

$$\sigma_Q^2 \approx \sum_i \left( \frac{\partial Q}{\partial x_i} \right)^2 \sigma_{x_i}^2$$
Error and Error Propagation

A Taylor series of a real function of a single variable, \( f(x) \), around a point \( x_0 \) is given by

\[
\begin{align*}
    f(x_0 + \Delta x) &= f(x_0) + f_x(x_0)\Delta x + \frac{1}{2!} f_{xx}(x_0)(\Delta x)^2 + \frac{1}{3!} f_{xxx}(x_0)(\Delta x)^3 + \ldots
\end{align*}
\]

where

\[
    f_{xx}(x_0) = \left[ \frac{d}{dx} \frac{d}{dx} f(x) \right]_{x=x_0}
\]

A Taylor series of a real function of two variables, \( f(x, y) \), is given by

\[
\begin{align*}
    f(x_0 + \Delta x, y_0 + \Delta y) &= \\
    &= f(x_0, y_0) + \left[ f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y \right] \\
    &+ \frac{1}{2!} \left[ f_{xx}(x_0, y_0)(\Delta x)^2 + 2f_{xy}(x_0, y_0)\Delta x\Delta y + f_{yy}(x_0, y_0)(\Delta y)^2 \right] \\
    &+ \frac{1}{3!} \left[ f_{xxx}(x_0, y_0)(\Delta x)^3 + 3f_{xxy}(x_0, y_0)(\Delta x)^2(\Delta y) + 3f_{xyy}(x_0, y_0)(\Delta x)(\Delta y)^2 + f_{yyy}(x_0, y_0)(\Delta y)^3 \right] + \ldots
\end{align*}
\]
Error Propagation

We determine the standard deviation of a quantity \( Q(x, y) \) that depends on two independent, random variables \( x \) and \( y \). A sample of \( N \) measurements of the variables yields pairs of values, \( x_i \) and \( y_i \), with \( i = 1, 2, \ldots, N \). For the sample one can compute the means, \( \bar{x} \) and \( \bar{y} \); the standard deviations, \( \sigma_x \) and \( \sigma_y \); and the values \( Q_i = Q(x_i, y_i) \). We assume that the scatter of the \( x_i \) and \( y_i \) about their means is small. We can then write a power-series expansion for the \( Q_i \) about the point \( (\bar{x}, \bar{y}) \), keeping only the first powers. Thus,

\[
Q_i = Q(x_i, y_i) \approx Q(\bar{x}, \bar{y}) + \frac{\partial Q}{\partial x} (x_i - \bar{x}) + \frac{\partial Q}{\partial y} (y_i - \bar{y}), \tag{E.36}
\]

where the partial derivatives are evaluated at \( x = \bar{x} \) and \( y = \bar{y} \).

By definition, the mean of \( Q \) is

\[
\bar{Q} = \frac{1}{N} \sum_{i=1}^{N} Q_i = \frac{1}{N} \sum_{i=1}^{N} Q(x_i, y_i)
\]

and the variance of \( Q \) is

\[
\sigma^2(Q) = \frac{1}{N} \sum_{i=1}^{N} (Q_i - \bar{Q})^2
\]

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## Error Propagation

where the partial derivatives are evaluated at \( x = \bar{x} \) and \( y = \bar{y} \). The mean value of \( Q_i \) is simply

\[
\bar{Q} \equiv \frac{1}{N} \sum_{i=1}^{N} Q_i = \frac{1}{N} \sum_{i=1}^{N} Q(x_i, y_i)
\]

\[
\approx \frac{1}{N} \sum_{i=1}^{N} \left[ Q(\bar{x}, \bar{y}) + \frac{\partial Q}{\partial x}(\bar{x}, \bar{y}) (x_i - \bar{x}) + \frac{\partial Q}{\partial y}(\bar{x}, \bar{y}) (y_i - \bar{y}) \right] = Q(\bar{x}, \bar{y}), \quad \text{E.36}
\]

since the sums of the \( x_i - \bar{x} \) and \( y_i - \bar{y} \) over all \( i \) in Eq. (E.36) are zero, by definition of the mean values. Thus, the mean value of \( Q \) is the value of the function \( Q(x, y) \) calculated at \( x = \bar{x} \) and \( y = \bar{y} \).
Chapter 3: Counting Statistics

Error and Error Propagation

The variance of the \( Q_i \) is given by

\[
\sigma_Q^2 = \frac{1}{N} \sum_{i=1}^{N} (Q_i - \bar{Q})^2. \tag{E.38}
\]

\[
Q_i = Q(x_i, y_i) \equiv Q(\bar{x}, \bar{y}) + \frac{\partial Q}{\partial x} (x_i - \bar{x}) + \frac{\partial Q}{\partial y} (y_i - \bar{y}), \tag{E.36}
\]

Applying Eq. (E.36) with \( \bar{Q} = Q(\bar{x}, \bar{y}) \), we find that

\[
\sigma_Q^2 = \frac{1}{N} \sum_{i=1}^{N} \left[ \frac{\partial Q}{\partial x} (x_i - \bar{x}) + \frac{\partial Q}{\partial y} (y_i - \bar{y}) \right]^2 \tag{E.39}
\]

\[
= \left( \frac{\partial Q}{\partial x} \right)^2 \frac{1}{N} \sum_{i=1}^{N} (x_i - \bar{x})^2 + \left( \frac{\partial Q}{\partial y} \right)^2 \frac{1}{N} \sum_{i=1}^{N} (y_i - \bar{y})^2 \]

\[
+ 2 \left( \frac{\partial Q}{\partial x} \right) \left( \frac{\partial Q}{\partial y} \right) \frac{1}{N} \sum_{i=1}^{N} (x_i - \bar{x})(y_i - \bar{y}). \tag{E.40}
\]

Variance of \( y \), or \( \sigma^2(y) \)

Covariance, \( \text{Cov}(x, y) \)
Error and Error Propagation

The last term, called the covariance of $x$ and $y$, vanishes for large $N$ if the values of $x$ and $y$ are uncorrelated. (The factors $y_i - \bar{y}$ and $x_i - \bar{x}$ are then just as likely to be positive as negative, and the covariance also decreases as $1/N$). We are left with the first two terms, involving the variances of the $x_i$ and $y_i$:

$$
\sigma_Q^2 = \left( \frac{\partial Q}{\partial x} \right)^2 \sigma_x^2 + \left( \frac{\partial Q}{\partial y} \right)^2 \sigma_y^2.
$$

(E.41)

This is one form of the error propagation formula, which is easily generalized to a function $Q$ of any number of independent random variables.

Assumptions ??
Error Propagation

Case 1: Sums or differences of counts – $u$ is the sum or difference of two random numbers representing counts measured in two independent experiments.

\[ u = x + y \quad \text{or} \quad u = x - y \]

\[ \sigma_u = \sqrt{\sigma_x^2 + \sigma_y^2} \]

Example: estimating the net counts from a sample.

net counts = total counts – background counts

or

\[ u = x - y \]
Error Propagation

Case 2: Multiplication or division by a constant

\[ u = Ax \]

\[ \sigma_u = A\sigma_x \]

Example: estimating the count rate,

\[ \text{counting rate} \equiv r = \frac{x}{t} \]

Assuming that the error in the measuring time is negligible, we get

\[ \sigma_r = \frac{\sigma_x}{t} \]
Error Propagation

Case 3: Multiplication or division of counts

\[ u = xy, \quad \frac{\partial u}{\partial x} = y \quad \frac{\partial u}{\partial y} = x \]

Using the equation

\[ \sigma_Q^2 \equiv \sum_i \left( \frac{\partial Q}{\partial x_i} \right)^2 \sigma_{x_i}^2 \]

One gets

\[ \sigma_u^2 = \frac{\partial u}{\partial x} \sigma_x^2 + \frac{\partial u}{\partial y} \sigma_y^2 = y \cdot \sigma_x^2 + x \cdot \sigma_y^2. \]

Therefore,

\[ \left( \frac{\sigma_u}{u} \right)^2 = \left( \frac{\sigma_x}{x} \right)^2 + \left( \frac{\sigma_y}{y} \right)^2 \]
Error Propagation in Net Count Rate Measurement

To find the standard deviation of \( r_n \), we apply Eq. (11.46) with \( Q = r_n, x = n_g, \) and \( \gamma = n_b \). From Eq. (11.49) we have \( \partial r_n / \partial n_g = 1/t_g \) and \( \partial r_n / \partial n_b = -1/t_b \). Thus, the standard deviation of the net count rate is given by

\[
\sigma_{nr} = \sqrt{\frac{\sigma^2_g}{t^2_g} + \frac{\sigma^2_b}{t^2_b}} = \sqrt{\sigma^2_{gr} + \sigma^2_{br}}.
\] (11.50)

Here \( \sigma_g \) and \( \sigma_b \) are the standard deviations of the numbers of gross and background counts, and \( \sigma_{gr} \) and \( \sigma_{br} \) are the standard deviations of the gross and background count rates. Equation (11.50) expresses the well-known result for the standard deviation of the sum or difference of two Poisson or normally distributed random variables. Using \( n_g \) and \( n_b \) as the best estimates of the means of the gross and background distributions and assuming that the numbers of counts obey Poisson statistics, we have \( \sigma^2_g = n_g \) and \( \sigma^2_b = n_b \). Therefore, the last equation can be written

\[
\sigma_{nr} = \sqrt{\frac{n_g}{t^2_g} + \frac{n_b}{t^2_b}} = \sqrt{\frac{r_g}{t_g} + \frac{r_b}{t_b}},
\] (11.51)
Error Propagation in Net Count Rate Measurement

Example

A long-lived radioactive sample is placed in a counter for 10 min, and 1426 counts are registered. The sample is then removed, and 2561 background counts are observed in 90 min. (a) What is the net count rate of the sample and its standard deviation? (b) If the counter efficiency with the sample present is 28%, what is the activity of the sample and its standard deviation in Bq? (c) Without repeating the background measurement, how long would the sample have to be counted in order to obtain the net count rate to within ±5% of its true value with 95% confidence? (d) Would the time in (c) also be sufficient to ensure that the activity is known to within ±5% with 95% confidence?

Turner, pp. 324.
Error Propagation in Net Count Rate Measurement

(a) What is the net count rate of the sample and its standard deviation?

Solution

(a) We have \( n_g = 1426 \), \( t_g = 10 \text{ min} \), \( n_b = 2561 \), and \( t_b = 90 \text{ min} \). The gross and background count rates are \( r_g = \frac{1426}{10} = 142.6 \text{ cpm} \) and \( r_b = \frac{2561}{90} = 28.5 \text{ cpm} \).

Therefore, the net count rate is \( r_n = 142.6 - 28.5 = 114 \text{ cpm} \). The standard deviation can be found from either of the expressions in (11.51). Using the first (which does not depend on the calculated values, \( r_g \) and \( r_b \)), we find

\[
\sigma_{nr} = \sqrt{\frac{n_g}{t_g^2} + \frac{n_b}{t_b^2}} = \sqrt{\frac{1426}{(10 \text{ min})^2} + \frac{2561}{(90 \text{ min})^2}} = 3.82 \text{ min}^{-1} = 3.82 \text{ cpm}. \tag{11.52}
\]

\[
\sigma_{nr} = \sqrt{\frac{r_g}{t_g^2} + \frac{r_b}{t_b^2}} = \sqrt{\frac{r_g}{t_g} + \frac{r_b}{t_b}}.
\]
Error Propagation in Net Count Rate Measurement

(b) If the counter efficiency with the sample present is 28%, what is the activity of the sample and its standard deviation in Bq? (c) Without repeating the background

Solution:

(b) Since the counter efficiency is $\epsilon = 0.28$, the inferred activity of the sample is $A = r_n/\epsilon = (114 \text{ min}^{-1})/0.28 = 407 \text{ dpm} = 6.78 \text{ Bq}$. The standard deviation of the activity is $\sigma_{nr}/\epsilon = (3.82 \text{ min}^{-1})/0.28 = 13.6 \text{ dpm} = 0.227 \text{ Bq}$.

\[
u = Ax
\]

\[
\sigma_u = A\sigma_x
\]
Error Propagation in Net Count Rate Measurement

(c) Without repeating the background measurement, how long would the sample have to be counted in order to obtain the net count rate to within ±5% of its true value with 95% confidence? (d) Would the time in (c) also be sufficient to ensure that the activity is known to within ±5% with 95% confidence?

Solution:

(c) A 5% uncertainty in the net count rate is 0.05\(r_n = 0.05 \times 114 = 5.70\) cpm. For the true net count rate to be within this range of the mean at the 95% confidence level means that 5.70 cpm = 1.96\(\sigma_{nr}\) (Table 11.2), or that \(\sigma_{nr} = 2.91\) cpm. Using the second expression in (11.51) with the background rate as before (since we do not yet know the new value of \(n_g\)), we write

\[
\sigma_{nr} = 2.91 \text{ min}^{-1} = \sqrt{\frac{142.6 \text{ min}^{-1}}{t_g} + \frac{28.5 \text{ min}^{-1}}{90 \text{ min}}}. 
\]

Solving, we find that \(t_g = 17.5\) min.

(d) Yes. The relative uncertainties remain the same and scale according to the efficiency. If the efficiency were larger and the counting times remained the same, then a larger number of counts and less statistical uncertainty would result.
The chance of the measured net count rate to fall within +5% of its true value is therefore 95%.

If we assume that the measured net count rate of 114 cpm is close enough to the true net count rate, then to ensure there is 95% chance that the measured net count rate would fall within +5% of its true value, we need

$$114 (cpm) \times 5\% = 1.96 \times \sigma_{nr}.$$ 

Remember that

$$\sigma_{nr} = \sqrt{\frac{n_g}{t_g^2} + \frac{n_b}{t_b^2}} = \sqrt{\frac{r_g}{t_g} + \frac{r_b}{t_b}},$$

then

$$5.71 (cpm) = 1.96 \cdot \sqrt{\frac{r_g}{t_g} + \frac{r_b}{t_b}} = 1.96 \sqrt{\frac{r_b + 114 (cpm)}{t_g} + \frac{r_b}{t_b}}, \quad \text{so} \quad t_g = 17.5 \ (min)$$
Error Propagation

Case 5: Combination of independent measurements with unequal errors

If N independent measurements of the same quantity have been carried out and not all the measurements have the same precision, what is the best way to estimate the best estimate of the mean value of the quantity to be measured?

The best estimate of the quantity, \(\langle x \rangle\), can be achieved by the weighted average

\[
\langle x \rangle = \frac{\sum_{i=1}^{N} a_i x_i}{\sum_{i=1}^{N} a_i}
\]

How to assign the weighting factors \(a_i\)'s?
Error Propagation

Let each individual measurement $x_i$ be given a weighting factor $a_i$ and the best value $\langle x \rangle$ computed from the linear combination

$$\langle x \rangle = \frac{\sum_{i=1}^{N} a_i x_i}{\sum_{i=1}^{N} a_i}$$

(3.45)

We now seek a criterion by which the weighting factors $a_i$ should be chosen in order to minimize the expected error in $\langle x \rangle$.

For brevity, we write

$$\alpha \equiv \sum_{i=1}^{N} a_i$$

so that

$$\langle x \rangle = \frac{1}{\alpha} \sum_{i=1}^{N} a_i x_i$$

Now apply the error propagation formula [Eq. (3.37)] to this case:

$$\sigma_{\langle x \rangle}^2 = \sum_{i=1}^{N} \left( \frac{\partial \langle x \rangle}{\partial x_i} \right)^2 \sigma_{x_i}^2$$

Knoll, p. 91.

NPRE 441, Principles of Radiation Protection, Spring 2021
Error Propagation

Now apply the error propagation formula [Eq. (3.37)] to this case:

\[
\sigma_{(x)}^2 = \sum_{i=1}^{N} \left( \frac{\partial \langle x \rangle}{\partial x_i} \right)^2 \sigma_{x_i}^2 = \frac{\sum_{i=1}^{N} a_i x_i}{\sum_{i=1}^{N} a_i} \Rightarrow \langle x \rangle = \frac{\sum_{i=1}^{N} a_i x_i}{\sum_{i=1}^{N} a_i} \quad \sigma_{x_i}^2 = \frac{\sum_{i=1}^{N} a_i^2 \sigma_{x_i}^2}{\sum_{i=1}^{N} a_i^2} \cdot
\]

\[
\sigma_{(x)}^2 = \frac{\beta}{\alpha^2}
\]

(3.46)

where

\[
\alpha \equiv \sum_{i=1}^{N} a_i \quad \beta \equiv \sum_{i=1}^{N} a_i^2 \sigma_{x_i}^2
\]

In order to minimize \( \sigma_{(x)} \), we must minimize \( \sigma_{(x)}^2 \) from Eq. (3.46) with respect to a typical weighting factor \( a_j \):

\[
0 = \frac{\partial \sigma_{(x)}^2}{\partial a_j} = \frac{\alpha^2 \frac{\partial \beta}{\partial a_j} - 2 \alpha \beta \frac{\partial \alpha}{\partial a_j}}{\alpha^4}
\]

(3.47)
Error Propagation

\[ a_j = \frac{\beta}{\sigma_{x_j}^2} \]

Putting this into the definition of \( \beta \), we obtain

\[ \beta = \sum_{i=1}^{N} a_i^2 \sigma_{x_i}^2 = \sum_{i=1}^{N} \left( \frac{\beta}{\sigma_{x_i}^2} \right)^2 \sigma_{x_i}^2 \]

or

\[ \beta = \left( \sum_{i=1}^{N} \frac{1}{\sigma_{x_i}^2} \right)^{-1} \]

Therefore, the proper choice for the normalized weighting coefficient for \( x_j \), is

\[ a_j = \frac{1}{\sigma_{x_j}^2} \left( \sum_{i=1}^{N} \frac{1}{\sigma_{x_i}^2} \right)^{-1} \]

(3.50)

We therefore see that each data point should be weighted inversely as the square of its own error.
Error Propagation

\[
\langle x \rangle = \frac{\sum_{i=1}^{N} a_i x_i}{\sum_{i=1}^{N} a_i}
\]

Therefore, the proper choice for the normalized weighting coefficient for \( x_j \), is

\[
a_j = \frac{1}{\sigma_{x_j}^2} \left( \sum_{i=1}^{N} \frac{1}{\sigma_{x_i}^2} \right)^{-1}
\]

(3.50)

We therefore see that each data point should be weighted inversely as the square of its own error.

\[
\frac{1}{\sigma_{x_i}^2} \sim \text{creditability of the measurement } x_i.
\]

\[
a_i = \frac{1}{\sigma_{x_i}^2} \left/ \left( \sum_{i=1}^{N} \frac{1}{\sigma_{x_i}^2} \right) \right. \sim \text{relative creditability of the measurement } x_i.
\]
Optimization of Counting Experiments

Case 6: Measuring the net count rate from a long-lived radioisotope.

\[ S \equiv \text{counting rate due to the source alone without background} \]
\[ B \equiv \text{counting rate due to background} \]

The measurement of \( S \) is normally carried out by counting the source plus background (at an average rate of \( S + B \)) for a time \( T_{S+B} \) and then counting background alone for a time \( T_B \). The net rate due to the source alone is then

\[ S = \frac{N_1}{T_{S+B}} - \frac{N_2}{T_B} \quad (2) \]

where \( N_1 \) and \( N_2 \) are the total counts in each measurement.

If the total measurement \( T = T_{S+B} + T_B \) is fixed, how to minimize the statistical error on the measured net count rate?
Error Propagation in Net Count Rate Measurement

To find the standard deviation of $r_n$, we apply Eq. (11.46) with $Q = r_n$, $x = n_g$, and $\gamma = n_b$. From Eq. (11.49) we have $\partial r_n/\partial n_g = l/t_g$ and $\partial r_n/\partial n_b = -1/t_b$. Thus, the standard deviation of the net count rate is given by

$$\sigma_{nr} = \sqrt{\frac{\sigma^2_g}{t_g^2} + \frac{\sigma^2_b}{t_b^2}} = \sqrt{\sigma^2_{gr} + \sigma^2_{br}}.$$  

(11.50)

Here $\sigma_g$ and $\sigma_b$ are the standard deviations of the numbers of gross and background counts, and $\sigma_{gr}$ and $\sigma_{br}$ are the standard deviations of the gross and background count rates. Equation (11.50) expresses the well-known result for the standard deviation of the sum or difference of two Poisson or normally distributed random variables. Using $n_g$ and $n_b$ as the best estimates of the means of the gross and background distributions and assuming that the numbers of counts obey Poisson statistics, we have $\sigma^2_g = n_g$ and $\sigma^2_b = n_b$. Therefore, the last equation can be written

$$\sigma_{nr} = \sqrt{\frac{n_g}{t_g^2} + \frac{n_b}{t_b^2}} = \sqrt{\frac{r_g}{t_g} + \frac{r_b}{t_b}},$$  

(11.51)
Limits of Detectability

For a counting system, it is useful to set a detection limit. That is, the amount of activity can be detected reliably.

The basic procedure could be

1. Setting a certain confidence level – the probability that a decision (on whether or not a source is present) is correct.

2. Define a quantity based on which the decision can be made. In the source counting case, it is the net count per unit time

\[ n_S = n_g - n_b \]

where

- \( n_S \) : net counts
- \( n_g \) : gross counts
- \( n_b \) : background counts

3. Finding a critical level, \( L_c \). If \( n_s \) exceeds \( L_c \), we assume source activity is present, otherwise we assume that the source does not contain activity.
False Positive and False Negative Errors

Due to the statistical fluctuation on the counts measured within a given time $t$, there will be

(1) many instances in which a positive $n_S$ is above the critical level even for samples with no activity, which leads to the false positive.

(2) and similarly, measured counts is lower than the critical level even when the source contains non-zero activity, which leads to the false negative.

![Diagram](image)

**Figure 3.14** The distributions expected for the net counts $N_S$ for the cases of (a) no activity present, and (b) a real activity present. $L_C$ represents the critical level or "trigger point" of the counting system.
False Positive and False Negative Errors

Type I error (false positive) and Type II error (false negative) are two types of errors that carry different implications.

False positive $\leftrightarrow$ minimum significant measured activity
False negative $\leftrightarrow$ minimum detectable true activity

![Diagram showing false positive and false negative errors]

- **False Positive**:
  - **NO REAL ACTIVITY**
  - Want to set $L_C$ high enough to minimize false positives

- **False Negative**:
  - **ACTIVITY PRESENT**
  - Want to set $N_D$ high enough to minimize false negatives
False Positive Rate and Minimum Significant Net Count Rate – An Example

Example
A sample, counted for 10 min, registers 530 gross counts. A 30-min background reading gives 1500 counts. (a) Does the sample have activity? (b) Without changing the counting times, what minimum number of gross counts can be used as a decision level such that the risk of making a type-I error is no greater than 0.050?

Turner, pp. 315-316
Solution

(a) The numbers of gross and background counts are \( n_g = 530 \) and \( n_b = 1500 \); the respective counting times are \( t_g = 10 \) min and \( t_b = 30 \) min. The gross and background count rates are \( r_g = n_g / t_g = 53 \) cpm and \( r_b = n_b / t_b = 50 \) cpm, giving a net count rate \( r_n = r_g - r_b = 3 \) cpm. The question of whether activity is present cannot be answered in an absolute sense from these measurements. The observed net rate could occur randomly with or without activity in the sample. We can, however, compute the probability that the result would occur randomly when we assume that the sample has no activity. To do this, we compare the net count rate with its estimated standard deviation \( \sigma_{nr} \), given by Eq. (11.51):

\[
\sigma_{nr} = \sqrt{\frac{r_g}{t_g} + \frac{r_b}{t_b}} = \sqrt{\frac{53}{10} + \frac{50}{30}} = 2.64 \text{ cpm.}
\] (11.64)

The observed net rate differs from 0 by \( 3/2.64 = 1.14 \) standard deviations. As found in Table 11.1, the area under the standard normal curve to the right of this value is 0.127. Assuming that the activity \( A \) is zero, as shown in Fig. 11.4, we conclude that an observation giving a net count rate greater than the observed \( r_n = 1.14\sigma_{nr} = 3 \) cpm would occur randomly with a probability of 0.127. This single set of measurements, gross and background, is thus consistent with the conclusion that the sample likely contains little or no activity. However, one does not know where the bell-shaped curve in Fig. 11.4 should be centered. Based on this single measurement, the most likely place is \( r_n = 3 \) cpm, with the sample activity corresponding to that value of the net count rate.
FIGURE 11.4. Probability density $P_n(r_n)$ for measurement of net count rate $r_n$ when net activity is present. See example in text. (Courtesy James S. Bogard.)
Possible conclusion #1: If there is no activity in the source, there will be 87% of chance of observing less than or equal to 3 cpm, the fact that we measured 3 cpm seems consistent with the assumption that there is no activity – So we can conclude that there is NO ACTIVITY in the source.

Possible conclusion #2: We don’t know where this bell-shaped distribution is. Based on the single measurement and the fact that we see 3 cpm, we may conclude that the source HAS ACTIVITY ...

Either decision would have risk ...

It would be easier to make the decision if we know what kind of risk would be associated with the decision and how much the risk that we could tolerate ...
False Positive Rate and Minimum Significant Net Count Rate – An Example

Example
A sample, counted for 10 min, registers 530 gross counts. A 30-min background reading gives 1500 counts. (a) Does the sample have activity? (b) Without changing the counting times, what minimum number of gross counts can be used as a decision level such that the risk of making a type-I error is no greater than 0.050?

\[ P(r_n) \]

\[ \sigma_{r_n} = \sqrt{\frac{r_g}{t_g} + \frac{r_b}{t_b}} \]

\[ r_n: \text{measured net count rate.} \]
False Positive Rate and Minimum Significant Net Count Rate – An Example

\[ r_1 = 1.65 \sqrt{\frac{r_g}{t_g} + \frac{r_b}{t_b}} = 1.65 \sqrt{\frac{r_1 + 50}{10} + \frac{50}{30}}, \]  

where the substitution \( r_g = r_1 + r_b \) has been made. This equation is quadratic in \( r_1 \).

After some manipulation, one finds that

\[ r_1^2 - 0.272r_1 - 18.2 = 0. \]  

The solution is \( r_1 = 4.40 \text{ cpm} \). The corresponding gross count rate is \( r_g = r_1 + r_b = 4.40 + 50 = 54.4 \text{ cpm} \), and so the critical number of gross counts is \( n_g = r_g t_g = 54.4 \text{ min}^{-1} \times (10 \text{ min}) = 544 \). Thus, a sample giving \( n_g > 544 \) (i.e., a minimum of 545 gross counts) can be reported as having significant activity, with a probability no greater than 0.05 of making a type-I error.
Minimum Significant Net Count Rate

**a)**

\[ P(N_S) \]

\[ \sigma_{N_S} \]

\[ L_C \]

**NO REAL ACTIVITY**

Want to set \( L_C \) high enough to minimize false positives

**b)**

\[ P(N_S) \]

\[ N_D \]

\[ \sigma_{N_D} \]

**ACTIVITY PRESENT**

Want to set \( N_D \) high enough to minimize false negatives

NPRE 441, Principles of Radiation Protection, Spring 2021
Minimum Significant Net Count Rate

Minimum significant measured net count rate ($r_1$) – the minimum measured net count rate that enables one to confirm the presence of activity and with a probability of false positive of less than a given threshold $\alpha$.

To derive the minimum significant measured net count rate ($r_1$), we write

$$r_1 = k_\alpha \sqrt{\sigma_{gr}^2 + \sigma_{br}^2} = k_\alpha \sqrt{\frac{r_1 + r_b}{t_g} + \frac{r_b}{t_b}}.$$

$\alpha$: maximum probability for false positive error
$k_\alpha$: number of standard deviations of the net count rate that gives a one-tail area (under a Gaussian distribution) equal to $\alpha$
$r_1$: the minimum significant measured net count rate
Minimum Significant Net Count Rate

To derive the minimum significant measured net count rate \( (r_1) \), we write

\[
r_1 = k_\alpha \sqrt{\sigma_{gr}^2 + \sigma_{br}^2} = k_\alpha \sqrt{\frac{r_1^2 + r_b^2}{t_g} + \frac{r_b}{t_b}}.
\]

Solving for \( r_1 \), we get the minimum significant measured net count rate \( (r_1) \) as

\[
r_1 = \frac{k_\alpha^2}{2t_g} + \frac{k_\alpha}{2} \sqrt{\frac{k_\alpha^2}{t_g^2} + 4r_b \left( \frac{t_g + t_b}{t_gt_b} \right)}.
\]
Minimum Significant Count Difference

When the gross and background only counting times are equal \( t \), we can derive the minimum significant count difference, \( \Delta_1 \), as

The minimum difference in the counts measured in both measurements (gross and background) that ensures the probability of having Type I error to be smaller than the threshold \( \alpha \).

\[
\Delta_1 = r_1 t = \frac{1}{2} k_\alpha^2 + \frac{1}{2} k_\alpha \sqrt{k_\alpha^2 + 8n_b},
\]

\[
= k_\alpha \sqrt{2n_b} \left( \frac{k_\alpha}{\sqrt{8n_b}} + \sqrt{1 + \frac{k_\alpha^2}{8n_b}} \right).
\]

In many instances, we have \( k_\alpha / \sqrt{n_b} \ll 1 \).

Then

\[
\Delta_1 \equiv k_\alpha \sqrt{2n_b},
\]
False Positive Rate and Minimum Significant Measured Count Difference

Often, the background can be measured accurately. The expected number of background counts $B$ in time $t$ is known.

In such case, if there is no source activity, the standard deviation of the net count is equal to $\sqrt{B}$. It follows that the minimum significant net count difference is

$$\Delta_1 = k_\alpha \sqrt{B} \quad \text{(Background accurately known)}.$$

$$\Delta_1 \equiv k_\alpha \sqrt{2n_b}, \quad k_\alpha / \sqrt{n_b} \ll 1.$$  

The minimum significant net count difference is lowered by a factor of 1.414 when the background is well known.
Minimum Significant Measured Activity

Consider that the measurements were done with a detector of efficiency $\varepsilon$, then the minimum significant measured activity is

$$A_I = \frac{\Delta_1}{\varepsilon t}.$$ 

If the measured net activity $A > A_I$, we state that the source contains activity, with the probability of false positive is $< \alpha$.

$$\Delta_1 = r_1 t = \frac{1}{2} k_\alpha^2 + \frac{1}{2} k_\alpha \sqrt{k_\alpha^2 + 8n_b},$$

$$= k_\alpha \sqrt{2n_b} \left( \frac{k_\alpha}{\sqrt{8n_b}} + \sqrt{1 + \frac{k_\alpha^2}{8n_b}} \right).$$
Minimum Significant Measured Activity

Example

A 10-min background measurement with a certain counter yields 410 counts. A sample is to be measured for activity by taking a gross count for 10 min. The maximum acceptable risk for making a type-I error is 0.05. The counter efficiency is such that 3.5 disintegrations in a sample result, on average, in one net count.

(a) Calculate the minimum significant net count difference and the minimum significant measured activity in Bq.

(b) How much error is made in (a) by using the approximate formula (11.72) in place of (11.69)?

(c) What is the decision level for type-I errors in terms of the number of gross counts in 10 min?
Minimum Significant Measured Activity

Example

A 10-min background measurement with a certain counter yields 410 counts. A sample is to be measured for activity by taking a gross count for 10 min. The maximum acceptable risk for making a type-I error is 0.05. The counter efficiency is such that 3.5 disintegrations in a sample result, on average, in one net count.

(a) Calculate the minimum significant net count difference and the minimum significant measured activity in Bq.

$$\Delta_1 = r_b t = \frac{1}{2} k^2_\alpha + \frac{1}{2} k_{\alpha}\sqrt{k^2_\alpha} + 8n_b,$$

Solution

(a) With equal counting times, $t_g = t_b = t = 10$ min, one can use Eq. (11.69) in place of the general expression (11.68). For $\alpha = 0.05$, $k_\alpha = 1.65$. With $n_b = 410$, we obtain

$$\Delta_1 = \frac{1}{2} (1.65)^2 + \frac{1}{2} (1.65)\sqrt{(1.65)^2 + 8(410)} = 48.6 = 49$$

(11.74)

for the minimum significant count difference in 10 min (rounded upward to the nearest integer). The counter efficiency is $\epsilon = 1/3.5 = 0.286$ dpm/cpm. It follows from Eq. (11.71) that the minimum significant measured activity is $A_I = 48.6/(0.286 \times 10 \text{ min}) = 17.0 \text{ dpm} = 0.283 \text{ Bq}$. 

$$A_I = \frac{\Delta_1}{\epsilon t}.$$
Minimum Significant Measured Activity

(b) How much error is made in (a) by using the approximate formula (11.72) in place of (11.69)?

(c) What is the decision level for type-I errors in terms of the number of gross counts in 10 min?

Solution

(c) The decision level for gross counts in 10 min is \( n_1 = n_b + \Delta_1 = 459 \).

\[
\Delta_1 = r_1 t = \frac{1}{2} k_\alpha^2 + \frac{1}{2} k_\alpha \sqrt{k_\alpha^2 + 8n_b},
\]
Minimum Significant Measured Activity

The value $n_1 = 459$ in the last example can serve as a decision level for screening samples for the presence of activity by gross counting for 10 min. A sample showing $n_g < 459$ counts can be reported as having less than the “minimum significant measured activity,” $A_1 = 0.283$ Bq. A sample showing $n_g \geq 459$ counts can be reported as having an activity $(n_g - n_b)/\epsilon t = (n_g - 410)/2.86$ dpm.
Minimum Significant Measured Activity

For samples having zero activity, the probability of making a type-I error is just equal to the value chosen for $\alpha$. For samples having activity, a type-I error cannot occur, by definition. Therefore, when one screens a large collection of samples, some with $A = 0$ and some with $A > 0$, the probability of making a type-I error with any given sample never exceeds $\alpha$.

**NO REAL ACTIVITY**
Want to set $L_C$ high enough to minimize false positives
False Negative and Minimum True Activity

Type-II Errors (False Negative) – wrongly conclude that no activity is present when there is actually activity in the source.

![Diagram showing the probability distribution of counts for different activities]

**ACTIVITY PRESENT**
Want to set $N_D$ high enough to minimize false negatives
False Negative and Minimum True Activity

Type-II Errors (False Negative) – wrongly conclude that “no active source is present” when there is an active source.

If we set a threshold level, $r_1$, what would be the minimum true source activity ($A_{II}$), so that the decision rule based on the threshold value $r_1$ can correctly detect the presence of the source with a probability of $\geq (1-\beta)$ or equivalently with the probability of making Type II error being $\leq \beta$?

$A > A_{II}$: probability of a false negative (type-II error) is less than $\beta$.

$A_{II}$ is called the minimum detectable true activity.
False Negative and Minimum True Count Rate, $r_2$

If we use $r_1$ as the critical decision threshold on measured count-rate to ensure that the probability of type-I error (false positive) is less than $\alpha$, what is the minimum true activity of the source ($A_{II}$) that would ensure the probability of type-II error (false negative) is less than $\beta$?

Assuming $r_2 \equiv A_{II} \cdot \varepsilon$, where $\varepsilon$ is the detection efficiency of the counting detector.

$r_n$: measured net count rate
False Negative and Minimum True Count Rate, $r_2$

To determine $r_2$, we would write

$$r_1 - r_2 = -k\beta \sqrt{\frac{r_g}{t_g} + \frac{r_b}{t_b}}. \quad (1)$$

We would further assume that $r_g = r_1 + r_b$ and substitute into the above equation

$$r_1 - r_2 = -k\beta \sqrt{\frac{r_1 + r_b}{t_g} + \frac{r_b}{t_b}}, \quad (2)$$

or

$$r_2 = r_1 + k\beta \sqrt{\frac{r_1}{t_g} + r_b \left( \frac{t_g + t_b}{t_g t_b} \right)}.$$  

Substituting for $r_1$ from Eq. (11.68),

$$r_1 = \frac{k^2}{2t_g} + \frac{k\alpha}{2} \sqrt{\frac{k^2}{t_g^2} + 4r_b \left( \frac{t_g + t_b}{t_g t_b} \right)}.$$
False Negative and Minimum True Count Rate, \( r_2 \)

we obtain

\[
r_2 = k_\alpha \left[ \frac{k_\alpha}{2t_g} + \frac{1}{2} \sqrt{\frac{k_\alpha^2}{t_g^2}} + 4r_b \left( \frac{t_g + t_b}{t_gt_b} \right) \right]
\]

\[+ k_\beta \left[ \frac{k_\alpha}{t_g} \left( \frac{k_\alpha}{2t_g} + \frac{1}{2} \sqrt{\frac{k_\alpha^2}{t_g^2}} + 4r_b \left( \frac{t_g + t_b}{t_gt_b} \right) \right) + r_b \left( \frac{t_g + t_b}{t_gt_b} \right) \right]^{1/2}.
\]

This general result gives the net rate that corresponds to the minimum detectable true activity for a given background rate \( r_b \) and arbitrary choices of \( \alpha, \beta, \) and the counting times.
False Negative and Minimum True Activity, $A_{II}$

From $r_2$, we can derive the minimum true activity, $A_{II} = \frac{r_2}{\varepsilon}$.

Decision threshold

Minimum mean net count rate, with given $\alpha, \beta$. 

$P_n(r_n)$

$A = 0$

$A_{II} > 0$

$0$

$r_1$

$r_2$

$r_n$: measured net count rate
False Negative and Minimum True Activity

Special case 1:

\[
\Delta_2 = r_2 t = \sqrt{2n_b} \left\{ k_\alpha \left[ \frac{k_\alpha}{\sqrt{8n_b}} + \sqrt{1 + \frac{k^2_\alpha}{8n_b}} \right] + k_\beta \left[ 1 + \frac{k^2_\alpha}{4n_b} + \frac{k_\alpha}{\sqrt{2n_b}} \sqrt{1 + \frac{k^2_\alpha}{8n_b}} \right]^{1/2} \right\}.
\]

With the help of Eq. (11.70), we can also write

\[
\Delta_2 = \Delta_1 + k_\beta \sqrt{2n_b} \left[ 1 + \frac{k^2_\alpha}{4n_b} + \frac{k_\alpha}{\sqrt{2n_b}} \sqrt{1 + \frac{k^2_\alpha}{8n_b}} \right]^{1/2}.
\]

The minimum detectable true activity is given by

\[
A_{II} = \frac{\Delta_2}{\epsilon t},
\]
False Negative and Minimum True Count Rate

Example
The counting arrangement ($\alpha = 0.05$, $\epsilon = 0.286$, $t_g = t_b = 10$ min, and $n_b = 410$) and critical gross count number $n_1 = 459$ from the last example are to be used to screen samples for activity. The maximum acceptable probability for making a type-II error is $\beta = 0.025$. (a) Calculate the minimum detectable true activity in Bq.
Limits of Detectability – An Example

(a) Calculate the minimum detectable true activity in Bq. (b) How much error is made by using the approximate formula (11.82) in place of the exact (11.79) or (11.80)

\[ \Delta_2 = \Delta_1 + k_\beta \sqrt{2n_b} \left[ 1 + \frac{k_\alpha^2}{4n_b} + \frac{k_\alpha}{\sqrt{2n_b}} \sqrt{1 + \frac{k_\alpha^2}{8n_b}} \right]^{1/2} \]  

(11.80)

\[ A_{II} = \frac{\Delta_2}{\epsilon t} \quad \Delta_2 : \text{Minimum detectable true count difference} \]  

(11.81)

Solution

(a) With equal gross and background counting times, we can use Eqs. (11.80) and (11.81) to find \( A_{II} \). For \( \beta = 0.025, k_\beta = 1.96 \) (Table 11.2). With \( \Delta_1 = 48.6 \) counts from the last example [Eq. (11.74)], Eq. (11.80) gives

\[ \Delta_2 = 48.6 + 1.96 \sqrt{2(410)} \left[ 1 + \frac{(1.65)^2}{4(410)} + \frac{1.65}{\sqrt{2(410)}} \sqrt{1 + \frac{(1.65)^2}{8(410)}} \right]^{1/2} = 106 \]

net counts. The minimum true detectable activity is, from Eq. (11.81),

\[ A_{II} = \frac{106}{0.286 \times 10 \text{ min}} = 37.1 \text{ dpm} = 0.618 \text{ Bq.} \]  

(11.87)
Limits of Detectability – An Example

This example illustrates how a protocol can be set up for reporting activity in a series of samples that are otherwise identical. As shown in Fig. 11.6, the decision level for a 10-min gross count is \( n_1 = 459 \), corresponding to the minimum significant count difference \( \Delta_1 = 49 \) and the minimum significant measured activity, \( A_1 = 0.283 \text{ Bq} \). A sample for which \( n_g < 459 \) is considered as having no reportable activity. When \( n_g \geq 459 \), a sample is reported as having an activity

\[
A = \frac{n_g - n_b}{\epsilon t} = \frac{n_g - 410}{(0.286)(600 \text{ s})} = \frac{n_g - 410}{172} \text{ Bq.} \quad (11.88)
\]

\( \alpha, \, n, \, \Delta, \, \gamma, \, A_\alpha \)

\( \beta, \, n_2, \, \Delta_2, \, \gamma_2, \, A_\beta \)
Limits of Detectability – An Example

\[ n_b = 410 \]
\[ n_1 = 459 \]
\[ n_2 = 516, \text{ minimum true count from the source.} \]

\[ \Delta_1 = 49 \]
\( \sqrt{(A_1 = 0.283 \text{ Bq})} \)

\[ \Delta_2 = 106 \]
\( \sqrt{(A_2 = 0.618 \text{ Bq})} \)

\[ n_g - n_b \]

...... and the chance of making type-I error is less than \( \alpha \).
Limits of Detectability – An Example

Note that $A$ will be greater than the minimum significant measured activity, $A_{1} = 0.283$ Bq. From part (a) in the last example, when $n_g = \Delta_2 + 410 = 516$, the reported value of the activity will be $A_{II} = 0.618$ Bq, the minimum detectable true activity. For a sample of unknown activity, the probability of making a type-I error does not exceed $\alpha = 0.05$. (If $A = 0$, the probability equals $\alpha$.) The probability of making a type-II error with a given sample does not exceed $\beta = 0.025$, as long as the activity is greater than $A_{II} = 0.618$ Bq. (If $A = A_{II}$, the probability equals $\beta$.) When $0 < A < A_{II}$, the probability for a type-II error is greater than $\beta$. 
False Negative and Minimum True Activity

From \( r_2 \), we can derive the minimum true activity, \( A_{II} = \frac{r_2}{\varepsilon} \).

\( P_n(r_n) \)

Decision threshold

Minimum mean net count rate, with given \( \alpha, \beta \).