Chapter 6: Counting Statistics

Table of Contents

Statistics of Radiation and Radiation Detection

Statistical nature of radiation and radiation interaction:

- ☞ How much energy will an 1 MeV photon lose in its next collision with an atomic electron?
- ☞ Will a 400keV photon penetrate a 2 mm lead shielding without interaction?
- ☞ When we use measured count-rate to estimate the activity of a source, and how certain are we on the estimation?

Exponential Radioactive Decay

Sample activity (A)

- ☞ True sample activity is never known.
- ☞ The best we can do is to repeat the counting process for a number of times and use the average as an indication of the sample activity – average number of decays in the sample per second.

$$A = A_0 e^{-\lambda t}$$

- ☞ The above equation can be interpreted by implying that the probability that an atom survives a time t without disintegration is

$$q = \text{probability of survival} = e^{-\lambda t}$$

$$and$$

$$p = \text{probability of decay} = 1\text{-}q = 1\text{-}e^{-\lambda t}$$

The actual number of decay events is fluctuating around the average value predicted by this equation.

Radioactive Disintegration – Bernoulli Process

Consider the radioactive disintegration process in a sample, it follows the following four conditions:

- It consists of *N* trials.
- Each trial has a binary outcome: success or failure (decay or not).
- The probability of success (decay) is a constant from trial to trial – all atoms have equal probability to decay.
- The trials are independent.

In statistics, these four conditions characterize a **Bernoulli** process.

Binomial Distribution

Given, p, N and t, what is the probability of observing n disintegrations within a time t?

☞ The number of ways to chose *n* atoms from a total of *N* atoms in the sample is

$$\binom{N}{n} = \frac{N!}{n!(N-n)!}$$

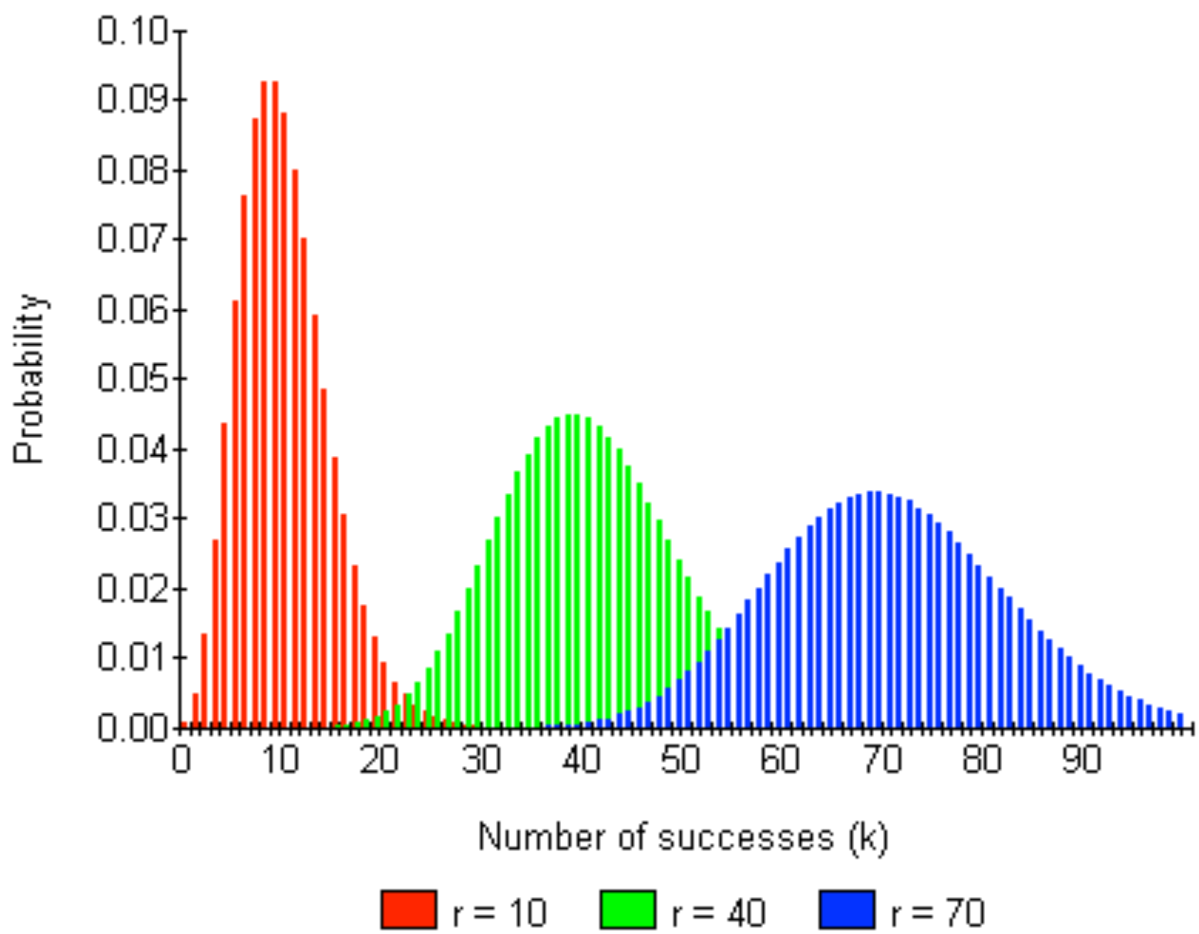
☞ So the probability of the *n* atoms chosen decayed during the time span *t* is

$$P_n = \binom{N}{n} p^n q^{N-n}$$

☞ The above equation describes the so-called **Binomial distribution**.

☞ What are the **mean** and **standard deviation** of a Binomial distribution?

Binomial Distribution



☞ What are the **mean** and **standard deviation** of a Binomial distribution?

Binomial Distribution

Mean

The mean value μ of the binomial distribution is defined by Eq. (11.15):

$$\mu \equiv \sum_{n=0}^{N} nP_n = \sum_{n=0}^{N} n \binom{N}{n} p^n q^{N-n}. \quad \text{(E.1)}$$

To evaluate this sum, we first use the binomial expansion to write, for an arbitrary (continuous) variable x,

$$(px + q)^N = \sum_{n=0}^{N} \binom{N}{n} p^n x^n q^{N-n} = \sum_{n=0}^{N} x^n P_n. \quad \text{(E.2)}$$

Differentiation with respect to x gives

$$Np(px + q)^{N-1} = \sum_{n=0}^{N} nx^{n-1} P_n. \quad \text{(E.3)}$$

Letting $x = 1$ and remembering that $p + q = 1$ gives

$$Np = \sum_{n=0}^{N} nP_n \equiv \mu. \quad \text{(E.4)}$$

Binomial Distribution

Standard Deviation

The variance is defined by Eq. (11.17):

$$\sigma^2 \equiv \sum_{n=0}^{N} (n - \mu)^2 P_n. \qquad \text{(E.5)}$$

This definition implies that

$$\sigma^2 = \sum_{n=0}^{N} (n^2 P_n - 2\mu n P_n + \mu^2 P_n) \qquad \text{(E.6)}$$

$$= \sum_{n=0}^{N} n^2 P_n - 2\mu \sum_{n=0}^{N} n P_n + \mu^2 \sum_{n=0}^{N} P_n. \qquad \text{(E.7)}$$

The first summation gives the expected value of n^2, the square of the number of disintegrations. From Eq. (E.4) it follows that the second term is $-2\mu^2$. The sum in the last term is unity [Eq. (11.14)]. Thus, we can write in place of Eq. (E.7)

$$\sigma^2 = \sum_{n=0}^{N} n^2 P_n - 2\mu^2 + \mu^2 = \sum_{n=0}^{N} n^2 P_n - \mu^2. \qquad \text{(E.8)}$$

We have previously evaluated μ [Eq. (E.4)]; it remains to find the sum involving n^2. To this end, we differentiate both sides of Eq. (E.3) with respect to x:

$$N(N-1)p^2(px+q)^{N-2} = \sum_{n=0}^{N} n(n-1)x^{n-2} P_n. \qquad \text{(E.9)}$$

Letting $x = 1$ with $p + q = 1$, as before, implies that

$$N(N-1)p^2 = \sum_{n=0}^{N} n(n-1) P_n \qquad \text{(E.10)}$$

$$= \sum_{n=0}^{N} n^2 P_n - \sum_{n=0}^{N} n P_n = \sum_{n=0}^{N} n^2 P_n - \mu. \qquad \text{(E.11)}$$

Thus,

$$\sum_{n=0}^{N} n^2 P_n = N(N-1)p^2 + \mu. \qquad \text{(E.12)}$$

Substituting this result into Eq. (E.8) and remembering that $\mu = Np$, we find that

$$\sigma^2 = N(N-1)p^2 + Np - N^2p^2 = Np(1-p) = Npq. \qquad \text{(E.13)}$$

The standard deviation of the binomial distribution is therefore

$$\sigma = \sqrt{Npq}. \qquad \text{(E.14)}$$

Binomial Distribution

For a binomial distribution, the **mean or the expectation** of the number of disintegration in time t is given by

$$\mu \equiv \sum_{n=0}^{N} n \cdot P_n = \sum_{n=0}^{N} n \cdot \binom{N}{n} p^n q^{N-n} = Np$$

and the fluctuation on the number of disintegrations is given by the **variance** or the **standard deviation** of the

$$\sigma^2 \equiv \sum_{n=0}^{N} (n-\mu)^2 \cdot P_n = Npq$$

$$and$$

$$\sigma \equiv \sqrt{\sum_{n=0}^{N} (n-\mu)^2 \cdot P_n} = \sqrt{Npq}$$

Binomial Distribution

Considering a realistic case, in which we use a detector to measure the number of counts and use the measured count rate to infer to the activity of the source.

Given (a) each disintegration yield one single particle and (b) the detection efficiency of the detector is ε, then

The prob. of detecting a count within a time t is

The probability of an atom disintegrates and results in a detected count is

$$p^* = \varepsilon p = \varepsilon(1 - e^{-\lambda t})$$

and the probability of an atom either does not disintegrate or the resultant particle is not detected is

$$q^* = 1 - \varepsilon p = 1 - \varepsilon + \varepsilon e^{-\lambda t}$$

Therefore, we can use the binomial distribution to describe the counting statistics as

Binomial Distribution

The **prob. of detecting *n* count within a time t** is

$$P_n^* = \binom{N}{n} (\varepsilon p)^n (1-\varepsilon p)^{N-n}$$

p : prob. of an atom disintergrates within a time t

ε : detection efficiency of the detector

The **mean number of detected counts** is

$$\mu^* \equiv \sum_{n=0}^{N} n \cdot P_n = \sum_{n=0}^{N} n \cdot \binom{N}{n} (p^*)^n (q^*)^{N-n} = \varepsilon N p$$

and the **variance** on the number of detected counts is

$$(\sigma^*)^2 \equiv \sum_{n=0}^{N} (n-\mu)^2 \cdot P_n^* = N p^* \cdot q^*$$

An Example Binomial Distribution

Example

More realistically, consider a ^{42}K source with an activity of 37 Bq (= 1 nCi). The source is placed in a counter, having an efficiency of 100%, and the numbers of counts in one-second intervals are registered.

(a) What is the mean disintegration rate?

(b) Calculate the standard deviation of the disintegration rate.

(c) What is the probability that exactly 40 counts will be observed in any second?

Binomial Distribution

For a binomial distribution, the **mean or the expectation** of the number of disintegration in time t is given by

$$\mu \equiv \sum_{n=0}^{N} n \cdot P_n = \sum_{n=0}^{N} n \cdot \binom{N}{n} p^n q^{N-n} = Np$$

and the fluctuation on the number of disintegrations is given by the **variance** or the **standard deviation** of the

$$\sigma^2 \equiv \sum_{n=0}^{N} (n-\mu)^2 \cdot P_n = Npq$$

$$and$$

$$\sigma \equiv \sqrt{\sum_{n=0}^{N} (n-\mu)^2 \cdot P_n} = \sqrt{Npq}$$

An Example Binomial Distribution

Solution

(a) The mean disintegration rate is the given activity, $r_d = 37\ s^{-1}$.

(b) The standard deviation of the disintegration rate is given by Eq. (11.18). We work with the time interval, $t = 1$ s. Since the decay constant is $\lambda = 0.0559\ h^{-1} = 1.55 \times 10^{-5}\ s^{-1}$, we have

$$q = e^{-\lambda t} = e^{-1.55 \times 10^{-5} \times 1} = 0.9999845 \qquad (11.26)$$

and $p = 1 - q = 0.0000155$.† The number of atoms present is

$$N = \frac{r_d}{\lambda} = \frac{37\ s^{-1}}{1.55 \times 10^{-5}\ s^{-1}} = 2.39 \times 10^6. \qquad (11.27)$$

From Eq. (11.18), we obtain for the standard deviation of the disintegration rate

$$\sigma_{dr} = \frac{\sqrt{Npq}}{t} = \frac{\sqrt{2.39 \times 10^6 \times 0.0000155 \times 0.9999845}}{1\ s} = 6.09\ s^{-1}, \qquad (11.28)$$

which is about 16% of the mean disintegration rate.

(c) The probability of observing exactly $n = 40$ counts in 1 s is given by Eq. (11.13). However, the factors quickly become unwieldy when N is not small (e.g., $69! = 1.71 \times 10^{98}$). For large N and small n, as we have here, we can write for the binomial coefficient

$$\binom{N}{n} \equiv \frac{N(N-1)\cdots(N-n+1)}{n!} \cong \frac{N^n}{n!}, \tag{11.29}$$

$$P_n = \binom{N}{n} p^n q^{N-n}$$

since each of the n factors in the numerator is negligibly different from N. Equation (11.13) then gives

$$P_{40} = \frac{(2.39 \times 10^6)^{40}}{40!}(0.0000155)^{40}(0.9999845)^{2.39 \times 10^6 - 40} \tag{11.30}$$

$$= \frac{(2.39)^{40}(10^{240})(0.0000155)^{40}(0.9999845)^{2.39 \times 10^6}}{40!}, \tag{11.31}$$

where $n = 40 \ll N$ has been dropped from the last exponent. The right-hand side can be conveniently evaluated with the help of logarithms. To reduce round-off errors, we use four decimal places:

$$\begin{aligned}
\log (2.39)^{40} &= 15.1359 \\
\log (10)^{240} &= 240.0000 \\
\log (0.0000155)^{40} &= -192.3867 \\
\log (0.9999845)^{2.39 \times 10^6} &= -16.0886 \\
-\log 40! &= \underline{-47.9116} \\
\log P_{40} &= -1.251 \, .
\end{aligned} \tag{11.32}$$

Thus, $P_{40} = 10^{-1.251} = 0.0561$.

Radioactive Disintegration – Bernoulli Process (Revisited)

Consider the radioactive disintegration process in a sample, it follows the following four conditions:

- It consists of N trials.
- Each trial has a binary outcome: success or failure (decay or not).
- The probability of success (decay) is a constant from trial to trial – all atoms have equal probability to decay.
- The trials are independent.

In statistics, these four conditions characterize a **Bernoulli** process.

Binomial Distribution (Revisited)

- The probability of exactly n decays out of a total of N atoms in the source is

$$P_n = \binom{N}{n} p^n q^{N-n}$$

For a binomial distribution, the **mean or the expectation** of the number of disintegration within the measurement period is given by

$$\mu \equiv \sum_{n=0}^{N} n \cdot P_n = \sum_{n=0}^{N} n \cdot \binom{N}{n} p^n q^{N-n} = Np$$

and the fluctuation on the number of disintegrations is quantified by the **variance**

$$\sigma^2 \equiv \sum_{n=0}^{N} (n-\mu)^2 \cdot P_n = Npq$$

Binomial Distribution – An Example

Consider a particle physics experiment – estimate the flux rate (number of particles per second across a unit solid angle) of a certain type of particle. Suppose the particles are coming from a point source.

Note that the detection efficiency of the detector is p=0.55, and the measurement has $S \cdot T = 1.9 \times 10^2\ cm^2 \cdot sec$.

If measurement did not register a single count, how do we estimate and report the flux rate (number of particles coming towards the detector surface per cm^2 per second) of the particle?

Radioactive Disintegration – Bernoulli Process

Consider the radioactive disintegration process in a sample, it follows the following four conditions:

- ☞ It consists of N trials.
- ☞ Each trial has a binary outcome: success or failure (decay or not).
- ☞ The probability of success (decay) is a constant from trial to trial – all atoms have equal probability to decay.
- ☞ The trials are independent.

In statistics, these four conditions characterize a **Bernoulli** process.

Binomial Distribution

Given, p, N and t, what is the probability of observing n disintegration within a time t?

- ☞ The number of ways to chose n atoms from N atoms in the sample is

$$\binom{N}{n} = \frac{N!}{n!(N-n)!}$$

- ☞ The probability of exactly n decays is

$$P_n = \binom{N}{n} p^n q^{N-n}$$

- ☞ The above probability function characterizes the so-called **Binomial** distribution.

Radioactive Disintegration – Bernoulli Process

- The Bernoulli Process and binomial distribution provide a nice statistic model for the decay of radioactive substances.
- In reality, we often encounter situations, in which p is very small and N is very large …
- In such cases, the statistical description of the decay process could be simplified …

Probability Distribution Function of Binomial Distribution for $N >> n \;\; and \;\; p \to 0$

For Binomial distribution,

$$P_n = \binom{N}{n} p^n q^{N-n}, \quad \text{where} \quad \binom{N}{n} = \frac{N!}{n!(N-n)!}$$

As in the previous example, for large N and small n, one can write

$\hookrightarrow p \to 0$

Binomial Expansion

$$(a+b)^n = \sum_{r=0}^{n} c_n^r \cdot a^{n-r} \cdot b^r = \sum_{r=0}^{n} \binom{n}{r} \cdot a^{n-r} \cdot b^r = \sum_{\gamma=0}^{n} \frac{n!}{(n-r)!\,r!} \cdot a^{n-r} \cdot b^r$$

$$\binom{N}{n} \equiv \frac{N(N-1)\cdots(N-n+1)}{n!} \cong \frac{N^n}{n!}, \tag{11.29}$$

$$q^{N-n} = 1 - Np + \frac{N(N-1)}{2!} p^2 - \cdots \tag{E.16}$$

Tylor Expansion

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \frac{x^0}{0!} + \frac{x'}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} \ldots$$

$$\cong 1 - Np + \frac{(Np)^2}{2!} - \cdots = e^{-Np}. \tag{E.17}$$

Substitution of Eqs. (11.29) and (E.17) into (11.13) gives

$$P_n = \frac{N^n}{n!} p^n e^{-Np} = \frac{(Np)^n}{n!} e^{-Np}, \tag{E.18}$$

Mean of Binomial Distribution for $N >> n \quad and \quad p \to 0$

The mean of binomial distribution is given by $\mu \equiv \sum_n n \cdot P_n$

where
$$P_n = \frac{N^n}{n!} p^n e^{-Np} = \frac{(Np)^n}{n!} e^{-Np},$$

therefore

$$\mu \equiv e^{-Np} \sum_{n=0}^{\infty} \frac{n(Np)^n}{n!} = e^{-Np} \sum_{n=1}^{\infty} \frac{n(Np)^n}{n!}$$

$$= e^{-Np} \sum_{n=1}^{\infty} \frac{(Np)^n}{(n-1)!} = e^{-Np} Np \sum_{n=1}^{\infty} \frac{(Np)^{n-1}}{(n-1)!}$$

$$= e^{-Np} Np \sum_{n=0}^{\infty} \frac{(Np)^n}{n!}$$

Mean of Binomial Distribution for $N >> n \quad and \;\; p \to 0$

To evaluate this sum, we first use the binomial expansion to write, for an arbitrary (continuous) variable x,

$$(px + q)^N = \sum_{n=0}^{N} \binom{N}{n} p^n x^n q^{N-n}$$

Therefore,

$$[1+(-p)]^N = 1 - Np + \frac{N(N-1)}{2!} p^2 - \cdots \qquad \text{(E.16)}$$

$$\cong 1 - Np + \frac{(Np)^2}{2!} - \cdots = e^{-Np}. \qquad \text{(E.17)}$$

Using E.17 and letting p' =-p, we have the following equation

$$\sum_0^\infty \frac{(Np')^n}{n!} = 1 + Np' + \frac{(Np')^2}{2!} + \cdots = e^{Np'}$$

Mean of Binomial Distribution for $N >> n \quad and \quad p \rightarrow 0$

Now go back to the mean of a Binomial distribution given by

$$\mu \equiv \sum_n n \cdot P_n = e^{-Np} Np \sum_{n=0}^{\infty} \frac{(Np)^n}{n!}$$

Since

$$\sum_0^{\infty} \frac{(Np)^n}{n!} = 1 + Np + \frac{(Np)^2}{2!} + \cdots = e^{Np},$$

then

$$\mu \equiv \sum_n n \cdot P_n = e^{-Np} Np \sum_{n=0}^{\infty} \frac{(Np)^n}{n!} = e^{-Np} Np e^{Np} = Np.$$

Standard Deviation and Variance of Binomial Distribution for $N >> n \quad and \quad p \to 0$

The variance of binomial distribution is given by

$$\sigma^2 \equiv \sum_{n=0}^{\infty} (n-\mu)^2 \cdot P_n = \left(\sum_{n=0}^{\infty} n^2 P_n\right) - \mu^2$$

Since

$$\sum_{n=0}^{\infty} n^2 P_n \equiv e^{-\mu} \sum_{n=0}^{\infty} \frac{n^2 \mu^n}{n!} = e^{-\mu} \sum_{n=1}^{\infty} \frac{n^2 \mu^n}{n!} \qquad (E.24)$$

$$\sum_{n=0}^{\infty} \frac{n\mu^n}{n!} = \sum_{n=1}^{\infty} \frac{n \cdot \mu^n}{n!} = \sum_{n=1}^{\infty} \frac{\mu^n}{(n-1)!}$$

$$= \mu \sum_{n=1}^{\infty} \frac{\mu^{n-1}}{(n-1)!} = \mu \sum_{n=0}^{\infty} \frac{\mu^n}{n!} = \mu e^{-\mu}$$

$$= e^{-\mu} \mu \sum_{n=1}^{\infty} \frac{n \mu^{n-1}}{(n-1)!} = e^{-\mu} \mu \sum_{n=0}^{\infty} \frac{(n+1)\mu^n}{n!} \qquad (E.25)$$

$$= e^{-\mu} \mu \sum_{n=0}^{\infty} \left(\frac{n\mu^n}{n!} + \frac{\mu^n}{n!}\right) = \mu(\mu+1) = \mu^2 + \mu. \qquad (E.26)$$

$$\sum_{n=0}^{\infty} \frac{\mu^n}{n!} = e^{\mu}$$

Substitute E.26 into the first equation, we have

$$\sigma^2 = \mu^2 + \mu - \mu^2 = \mu.$$

And the standard deviation is given by $\sigma = \sqrt{\mu}.$

Radioactive Disintegration – Bernoulli Process

Consider the radioactive disintegration process in a sample, it follows the following four conditions:

- It consists of N trials.
- Each trial has a binary outcome: success or failure (decay or not).
- The probability of success (decay) is a constant from trial to trial – all atoms have equal probability to decay.
- The trials are independent.

In statistics, these four conditions characterize a **Bernoulli** process.

What happens if $N >> n \quad and \quad p \rightarrow 0$?

Poisson Process

The counting statistics related to nuclear decay processes is often more conveniently described by the **Poisson** distribution, is related to situations that involves a collection of multiple trials that satisfy the following conditions:

1. The number of trials, N, is very large, e.g. $N>>1$.
2. Each trial is independent.
3. The probability that each single trial is successful is a constant and approaching zero, $p<<1$. So the number of successful trials is fluctuating around a finite number.

Binomial Distribution and Poisson Distribution

Binomial distribution

The probability of observing n successful trails out of a total of N independent trails:

$$P_n = \binom{N}{n} p^n q^{N-n}$$

mean of the observed number of successful trails:

$$\mu \equiv \sum_{n=0}^{N} n \cdot P_n = \sum_{n=0}^{N} n \cdot \binom{N}{n} p^n q^{N-n} = Np$$

Standard deviation:

$$Std(n) \equiv \sqrt{\sum_{n=0}^{N} (n-\mu)^2 \cdot P_n} = \sqrt{Npq}$$

Poisson distribution when N>>1, p<<1

$$P(n \mid \mu) = \frac{\mu^n}{n!} e^{-\mu}$$

Mean of n:

$$\mu(n) = \mu = N \cdot p$$

Standard deviation:

$$\sigma = \sqrt{\mu} = \sqrt{Np}$$

Gaussian distribution, If N is further increased, and p is further decreased

$$p(x \mid \mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Poisson Distribution

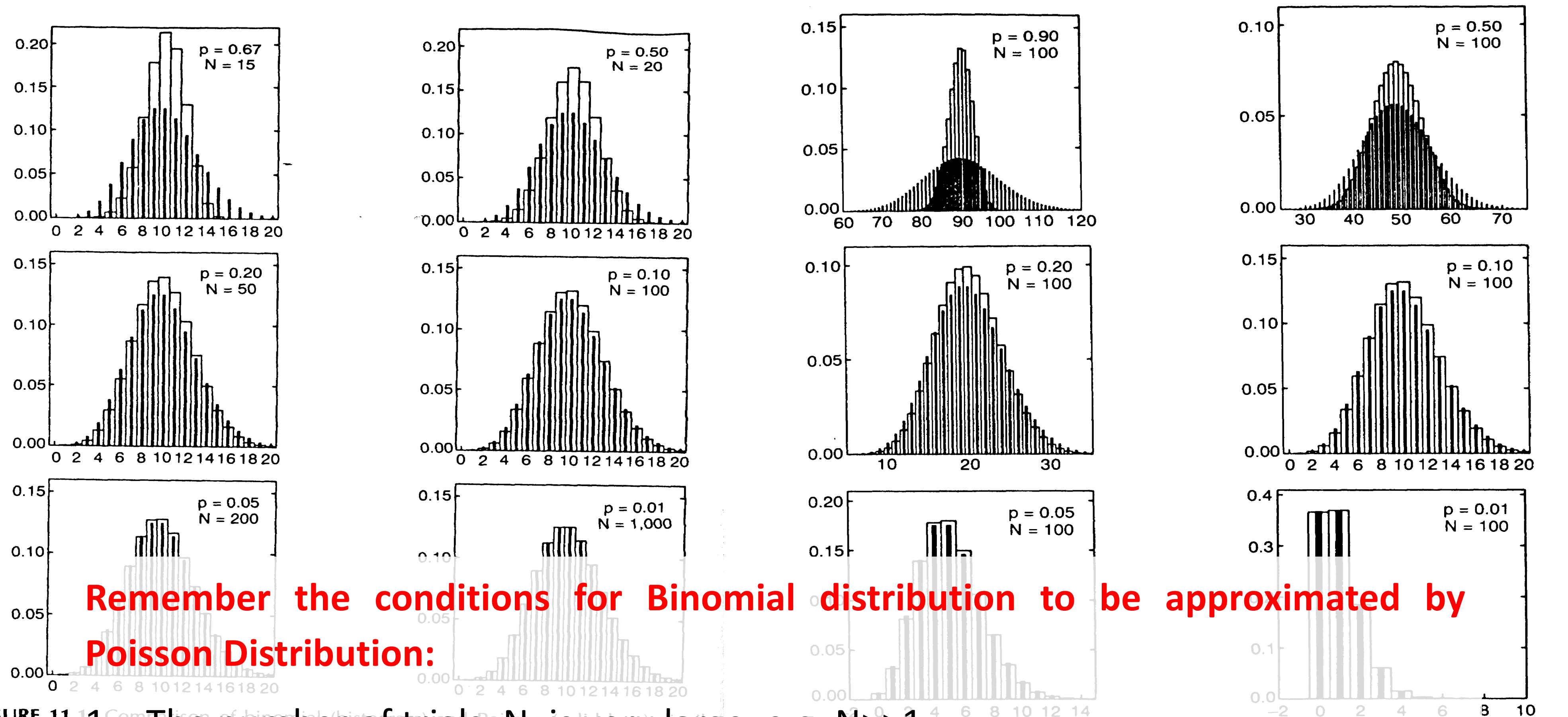


FIGURE 11.1 Comparison of binomial (histogram) and Poisson (solid bars) distributions, having the same mean, $\mu = 10$, but different values of the probability of success p and sample size N. The ordinate in each panel shows the probability P_n of exactly n successes, shown on the abscissa. With fixed μ, the Poisson distribution is the same throughout. (Courtesy James S. Bogard.)

FIGURE 11.2. Comparison of binomial (histogram) and Poisson (solid bars) distributions for fixed N and different p. The ordinate shows P_n and the abscissa, n. The mean of the two distributions in a given panel is the same. (Courtesy James S. Bogard.)

Remember the conditions for Binomial distribution to be approximated by Poisson Distribution:

1. The number of trials, N, is very large, e.g. N>>1.
2. Each trial is independent.
3. The probability that each single trial is successful is a constant and approaching zero, p<<1. So the number of successful trials is fluctuating around a finite number.

Poisson Distribution

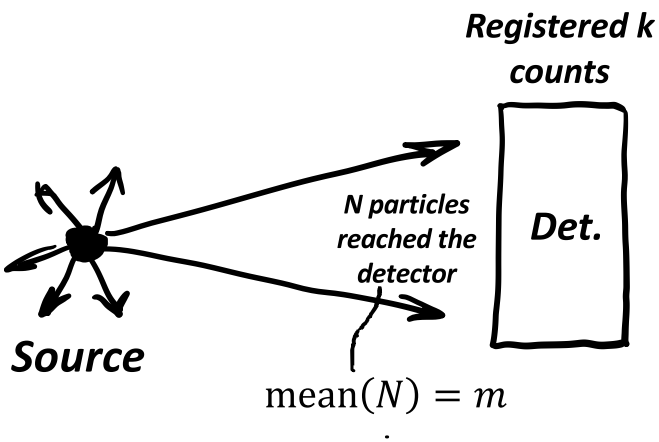
The probability of having *n* successful trials can be approximated with the Poisson distribution.

$$P(n \mid \mu) = \frac{\mu^n}{n!} e^{-\mu}$$

and the mean and the variance of number of successful trial are given by

$$Mean(n) = \mu = N \cdot p$$

$$Std(n) \equiv \sigma = \sqrt{\mu} = \sqrt{Np}$$



1. ***The detector covers 10% solid angle.***
2. ***detection efficiency: λ =55%.***
3. ***The measurement takes T = 30 min.***
4. ***N particles reached the detector.***
5. ***detected k=0 count.***

The prob. of having *N* particles reaching the detector would follow the Poisson distribution,

$$P(N) = \frac{m^N}{N!}e^{-m},$$

where m is the expected number of particles that reach the detector.

Once the *N* particles reached the detector, the number of particles detected would follow the Binomial distribution, so that the probability of detecting *k* particles is

$$P(k|N) = \binom{N}{k}\lambda^k(1-\lambda)^{N-K}.$$

Poisson Distribution

Therefore, the total probability of detecting k counts is

$$P(k) = \sum_{N=k}^{\infty} P(k|N)P(N|m)$$

$$= \sum_{N=k}^{\infty} \frac{N!}{(N-k)!k!} \lambda^k (1-\lambda)^{N-k} \cdot \frac{m^N}{N!} \mathrm{e}^{-m}$$

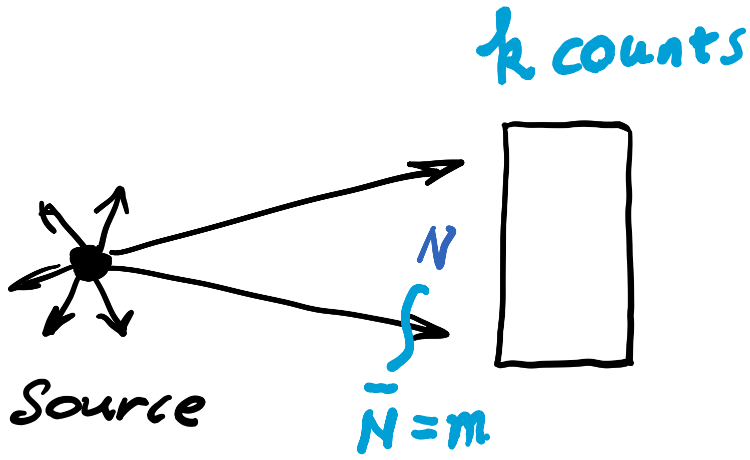
$$= \frac{(m\lambda)^k}{k!} \mathrm{e}^{-(m\lambda)}$$

If we would like to have 90% chance of detecting at least 1 particle, then we could write

$$1 - P(k=0) = \mathrm{e}^{-m\lambda} = 0.9,$$

then the mean number of particles reaching the detector during the 30 minute measurement should be

$$m = 4.2.$$



① *The detector covers 10% solid angle with respect to the point source.*

② *detection efficiency: λ =55%.*

③ *The measurement takes T = 30 min.*

④ *N particles reached the detector.*

⑤ *detected k=0 count.*

So finally, we can conclude that:

Because we did not record any count, we have 90% confidence to claim that the source strength (average number of particles emitted per second) should not exceed

$$A \leq \frac{4.2}{10\%} = 42 \ (particles\ per\ sec) = 42\ Bq$$

Binomial Distribution and Poisson Distribution

Binomial distribution

The probability of observing n successful trails out of a total of N independent trails:

$$P_n = \binom{N}{n} p^n q^{N-n}$$

mean of the observed number of successful trails:

$$\mu \equiv \sum_{n=0}^{N} n \cdot P_n = \sum_{n=0}^{N} n \cdot \binom{N}{n} p^n q^{N-n} = Np$$

Standard deviation:

$$Std(n) \equiv \sqrt{\sum_{n=0}^{N} (n-\mu)^2 \cdot P_n} = \sqrt{Npq}$$

Poisson distribution when N>>1, p<<1

$$P(n \mid \mu) = \frac{\mu^n}{n!} e^{-\mu}$$

Mean of n:

$$\mu(n) = \mu = N \cdot p$$

Standard deviation:

$$\sigma = \sqrt{\mu} = \sqrt{Np}$$

Effect of Fano Factor on Energy Resolution

→ ~3eV for silicon → 100 keV gives 30000 e-h pairs → what is the standard deviation associated with the number of e-h pairs per 100 keV energy deposition?

→ ~30eV for gas detectors → 100 keV gives 3000 e-ion pairs → ??

→ ~1 keV for NaI(Tl) → 100 keV only gives 100 photoelectrons → ??

→ The measured Fano factors: 0.143 for silicon, 0.129 for germanium, 0.1 for CdZnTe and ~0.1 for HgI_2.

→ In comparison, the Fano factors for gas and scintillators are ~1.

Semiconductor Detector Configurations

High-purity germanium (HPGe) detectors

- Supper-pure material available, for example 1 part in 10^{12}
- Depletion depth of >1cm → good detection volume.
- Requires cooling to liquid nitrogen temperature to reduce leakage current.

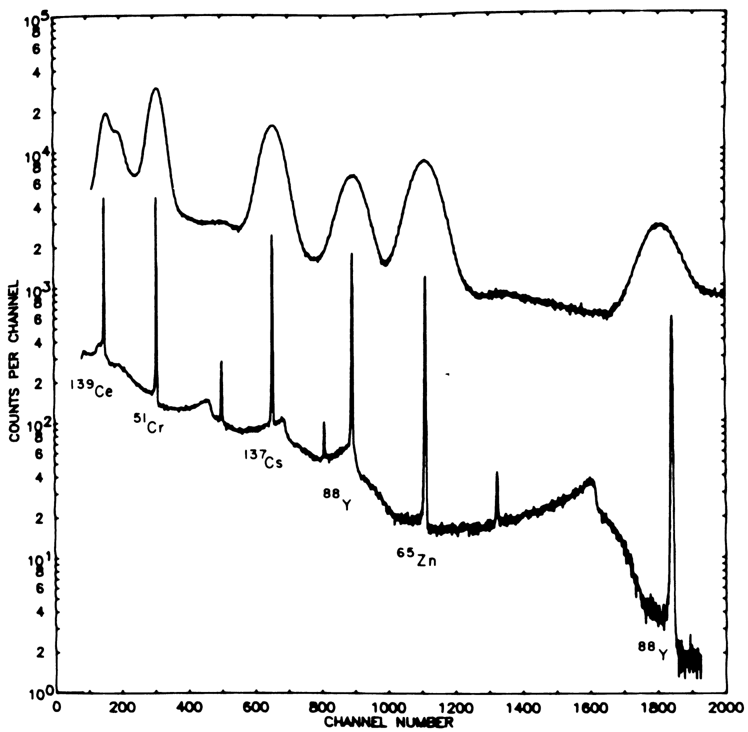


FIGURE 10.29. Comparison of gamma-ray spectra from a solution containing radionuclides as measured with a NaI scintillator (upper curve) and a Ge(Li) detector. [Reprinted with permission from *A Handbook of Radioactivity Measurements*, NCRP Report No. 58, p. 240, National Council on Radiation Protection and Measurements, Washington, D.C. (1978). Copyright 1978 National Council on Radiation Protection and Measurements.]

Signal Generation by Ionizing Radiation in Semiconductors

The Fano factor.

$$F \equiv \frac{\text{observed statistical variance}}{E/\epsilon}$$

→ For a given energy deposition E in the detector and a known energy ε required to create an e-h pair, the observed fluctuation in the number of charge carriers created is smaller than the one predicted by the Poisson statistics.

→ The measured Fano factors: 0.143 for silicon, 0.129 for germanium, 0.1 for CdZnTe and ~0.1 for HgI_2.

→ In comparison, the Fano factors for scintillators are ~1.

NORMAL DISTRIBUTION

We begin with Eq. (E.23) for the Poisson P_n and assume that μ is large. We also assume that the P_n are appreciably different from zero only over a range of values of n about the mean such that $|n - \mu| \ll \mu$. That is, the distribution of the P_n is relatively narrow about μ; and both μ and n are large. We change variables by writing $x = n - \mu$. Equation (E.23) can then be written

$$P_x = \frac{\mu^{\mu+x} e^{-\mu}}{(\mu + x)!} = \frac{\mu^{\mu} \mu^{x} e^{-\mu}}{\mu!(\mu + 1)(\mu + 2) \cdots (\mu + x)}, \tag{E.29}$$

with $|x| \ll \mu$. We can approximate the factorial term for large μ by means of the Stirling formula,

$$\mu! = \sqrt{2\pi\mu}\, \mu^{\mu} e^{-\mu}, \tag{E.30}$$

giving

$$P_x = \frac{\mu^x}{\sqrt{2\pi\mu}(\mu + 1)(\mu + 2) \cdots (\mu + x)} \tag{E.31}$$

$$= \frac{1}{\sqrt{2\pi\mu}\left(1 + \frac{1}{\mu}\right)\left(1 + \frac{2}{\mu}\right) \cdots \left(1 + \frac{x}{\mu}\right)}. \tag{E.32}$$

Since, for small y, $e^y \cong 1 + y$, the series of factors in the denominator can be rewritten (μ is large) to give

$$P_x = \frac{1}{\sqrt{2\pi\mu}\, e^{1/\mu} e^{2/\mu} \cdots e^{x/\mu}} = \frac{1}{\sqrt{2\pi\mu}} e^{-(1+2+\cdots+x)/\mu}. \tag{E.33}$$

The sum of the first x positive integers, as they appear in the exponent, is $x(1 + x)/2 = (x^2 + x)/2 \cong x^2/2$, where x has been neglected compared with x^2. Thus, we find that

$$P_x = \frac{1}{\sqrt{2\pi\mu}} e^{-x^2/2\mu}. \tag{E.34}$$

This function, which is symmetric in x, represents an approximation to the Poisson distribution. The normal distribution is obtained when we replace the Poisson standard deviation $\sqrt{\mu}$ by an independent parameter σ and let x be a continuous random variable with mean value μ (not necessarily zero). We then write for the probability density in x $(-\infty < x < \infty)$ the normal distribution

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/2\sigma^2}, \tag{E.35}$$

with $\sigma^2 > 0$. It can be shown that this density function is normalized (i.e., its integral over all x is unity) and that its mean and standard deviation are, respectively, μ and σ. The probability that the value of x lies between x and $x + dx$ is $f(x)\,dx$. Whereas the Poisson distribution has the single parameter μ, the normal distribution is characterized by the two independent parameters, μ and σ.

Poisson Distribution

The probability of having n successful trials can be approximated with the Poisson distribution.

$$P(n \mid \mu) = \frac{\mu^n}{n!} e^{-\mu}$$

and the mean and the variance of number of successful trial are given by

$$Mean(n) = \mu = N \cdot p$$

$$Std(n) \equiv \sigma = \sqrt{\mu} = \sqrt{Np}$$

The Gaussian (Normal) Distribution

As p (the prob. of an atom decay within t) is getting even smaller and N is getting larger, both Binomial and Poisson distributions are approaching an extremely useful form of distribution – the Gaussian distribution.

Gaussian distribution is defined for a continuous variable x

$$p(x \mid \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

but it is very useful for describing the counting fluctuation on discrete numbers.

Poisson Distribution

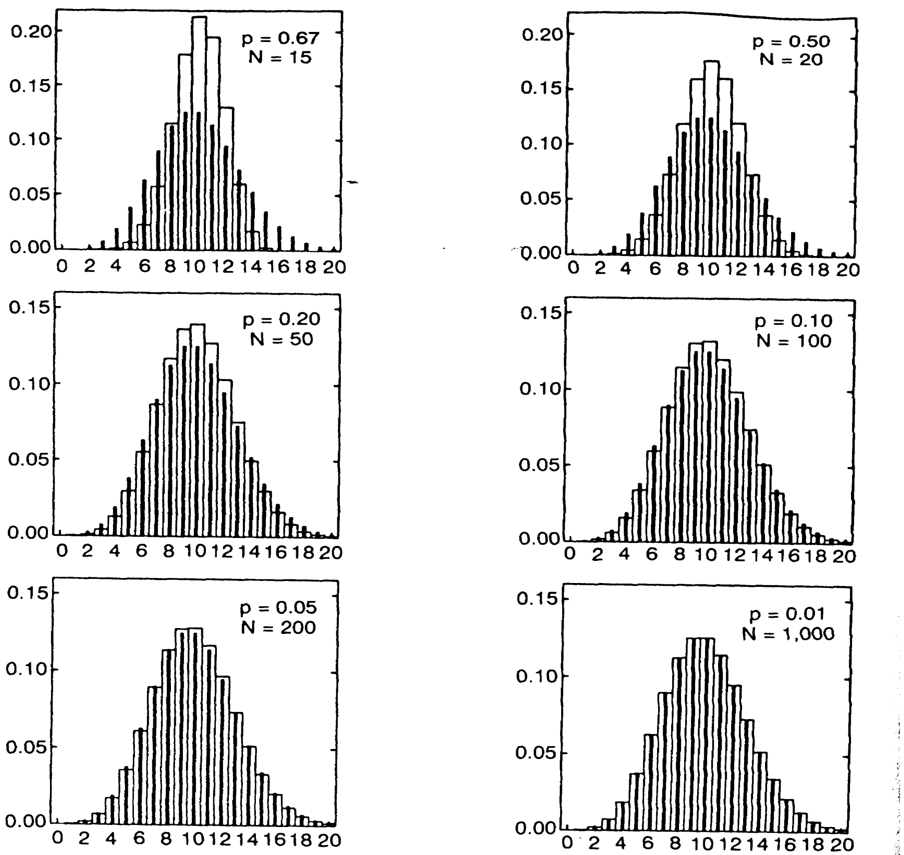


FIGURE 11.1. Comparison of binomial (histogram) and Poisson (solid bars) distribu tions, having the same mean, $\mu = 10$, but different values of the probability of succes p and sample size N. The ordinate in each panel shows the probability P_n of exactly n successes, shown on the abscissa. With fixed μ, the Poisson distribution is the same throughout. (Courtesy James S. Bogard.)

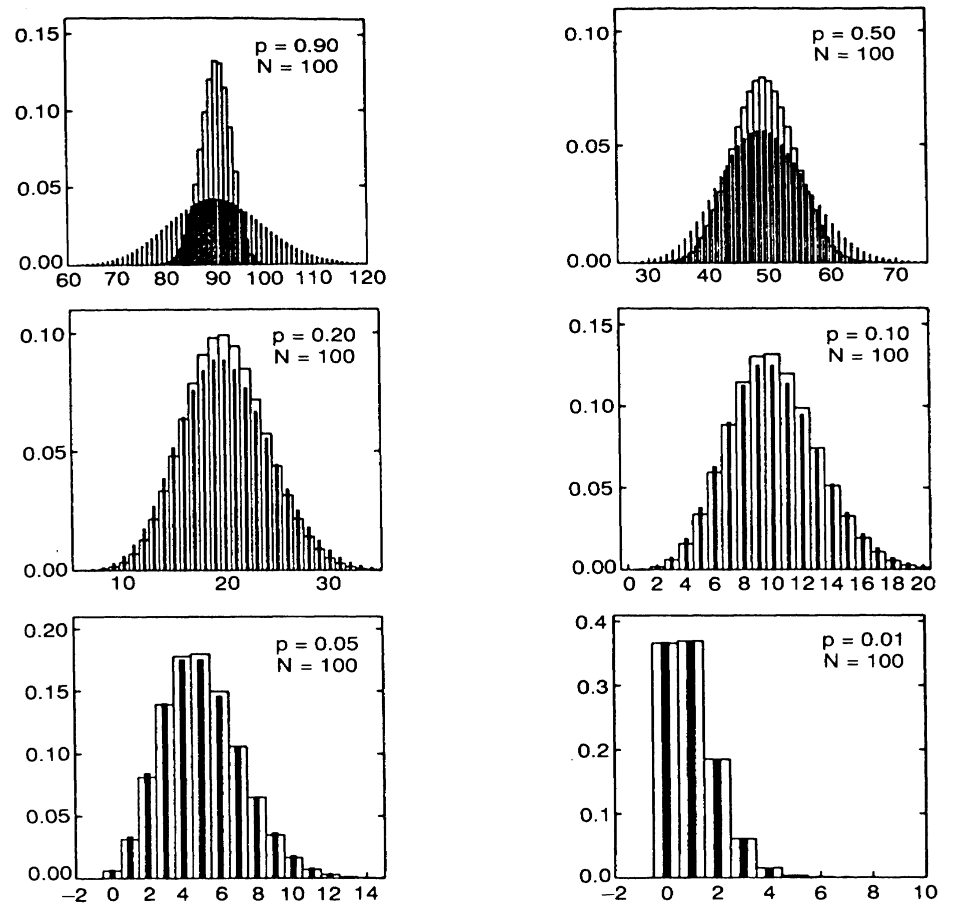


FIGURE 11.2. Comparison of binomial (histogram) and Poisson (solid bars) distributions for fixed N and different p. The ordinate shows P_n and the abscissa, n. The mean of the two distributions in a given panel is the same. (Courtesy James S. Bogard.)

The Gaussian (Normal) Distribution

Binomial and Poisson distributions practically match the normal distribution when μ >=30.

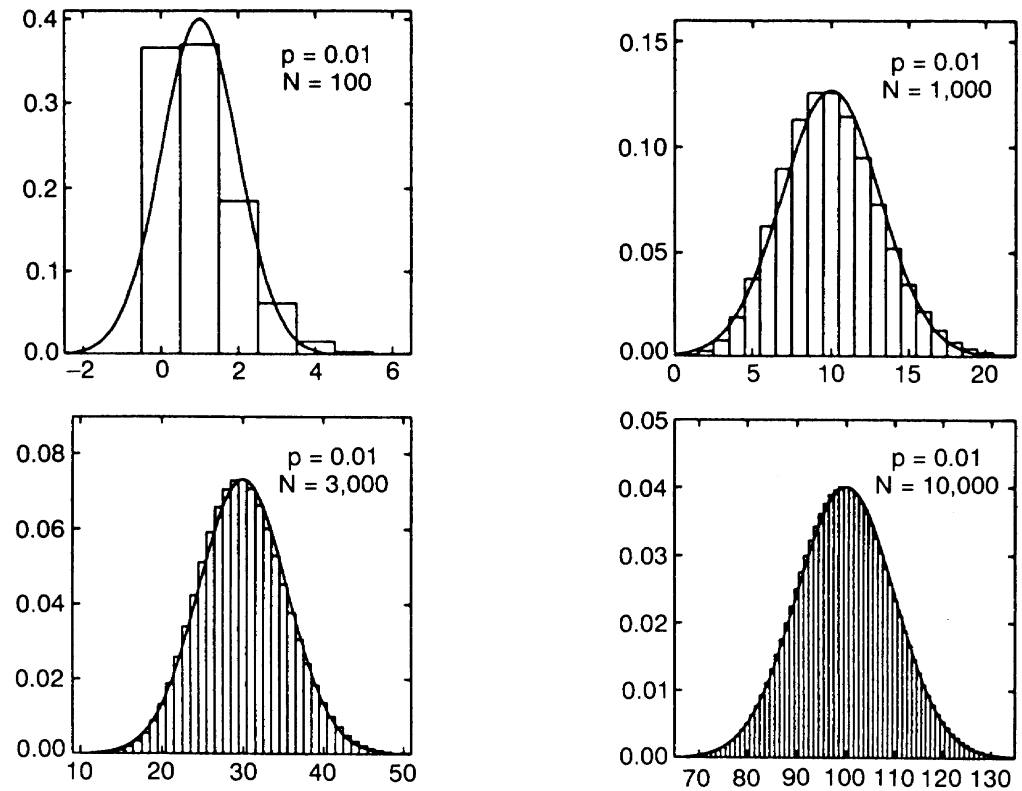


FIGURE 11.3. Comparison of binomial (histogram) and normal (solid line) distributions, having the same means and standard deviations. The ordinate in each panel gives the probability P_n for the former and the density $f(x)$ [Eq. (11.37)] for the latter, the abscissa giving n or x. (Courtesy James S. Bogard.)

Binomial Distribution and Poisson Distribution

Binomial distribution

The probability of observing n successful trails out of a total of N independent trails:

$$P_n = \binom{N}{n} p^n q^{N-n}$$

mean of the observed number of successful trails:

$$\mu \equiv \sum_{n=0}^{N} n \cdot P_n = \sum_{n=0}^{N} n \cdot \binom{N}{n} p^n q^{N-n} = Np$$

Standard deviation:

$$Std(n) \equiv \sqrt{\sum_{n=0}^{N} (n-\mu)^2 \cdot P_n} = \sqrt{Npq}$$

Poisson distribution when N>>1, p<<1

$$P(n \mid \mu) = \frac{\mu^n}{n!} e^{-\mu}$$

Mean of n:

$$\mu(n) = \mu = N \cdot p$$

Standard deviation:

$$\sigma = \sqrt{\mu} = \sqrt{Np}$$

Gaussian distribution, If N is further increased, and p is further decreased

$$p(x \mid \mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

The Gaussian (Normal) Distribution

Binomial and Poisson distributions practically match the normal distribution when μ >=30.

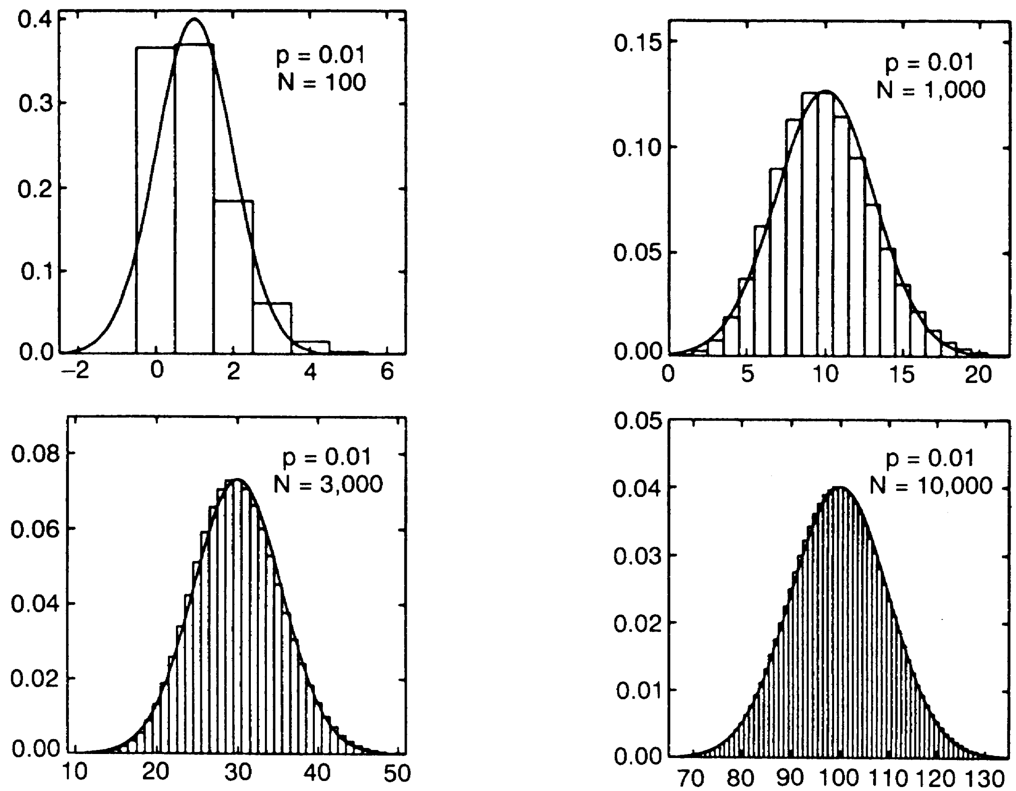


FIGURE 11.3. Comparison of binomial (histogram) and normal (solid line) distributions, having the same means and standard deviations. The ordinate in each panel gives the probability P_n for the former and the density $f(x)$ [Eq. (11.37)] for the latter, the abscissa giving n or x. (Courtesy James S. Bogard.)

The Gaussian (Normal) Distribution

For a variable, x, following the normal distribution, the probability that it takes a value between x and x is equal to the area under under the curve p(x) between these two values:

$$P(x_1 \leq x \leq x_2) = \frac{1}{\sqrt{2\pi\sigma}} \int_{x_1}^{x_2} e^{-(x-\mu)^2/2\sigma^2}\, dx.$$

Many common manipulations when carried out on counting data that were originally Gaussian distributed will produce derived values that also follow Gaussian shape:

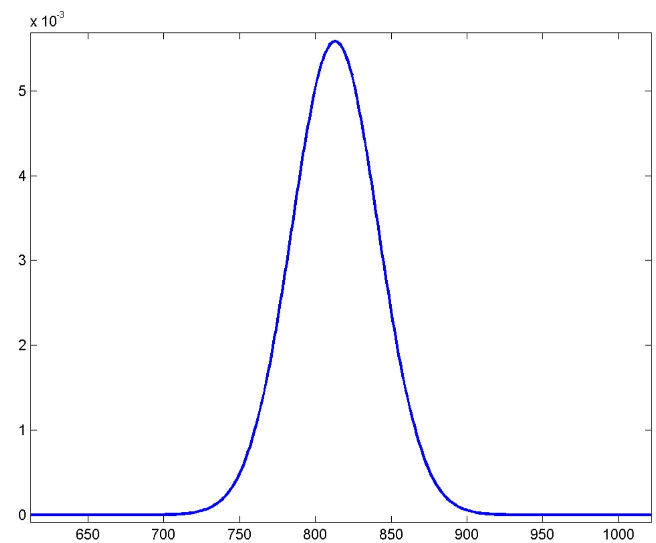
- ☞ Multiplying or dividing the data by a constant,
- ☞ Combining two Gaussian-distributed variables through addition, subtraction, or multiplication or,
- ☞ Calculating the average of a series of independent measurements.

The Gaussian (Normal) Distribution

Example

Repeated counts are made in 1-min intervals with a long-lived radioactive source. The observed mean value of the number of counts is 813, with a standard deviation of 28.5 counts. (a) What is the probability of observing 800 or fewer counts in a given minute? (b) What is the probability of observing 850 or more counts in 1 min? (c) What is the probability of observing 800 to 850 counts in a minute? (d) What is the symmetric range of values about the mean number of counts within which 90% of the 1-min observations are expected to fall?

$$P(x_1 \le x \le x_2) = \frac{1}{\sqrt{2\pi}\sigma} \int_{x_1}^{x_2} e^{-(x-\mu)^2/2\sigma^2}\, dx.$$



Turner, pp. 316-317.

Central Limit Theorem

The sum or average of a large number of independent random variables follows Gaussian (Normal) distribution.

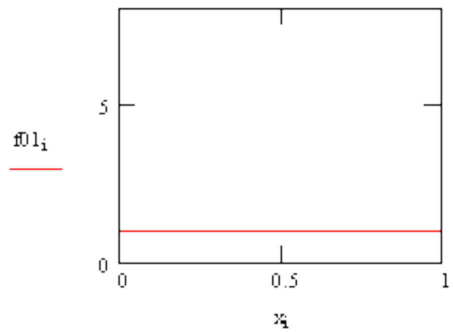
The distribution of an average tends to be NORMAL, even when the distribution of the underlying variables from which the average is computed is decidedly non-Normal.

Central Limit Theorem

Consider a series of independent and identically distributed (i.i.d.) random variables, x_1, x_2, …, x_n, whose probability density function are given by

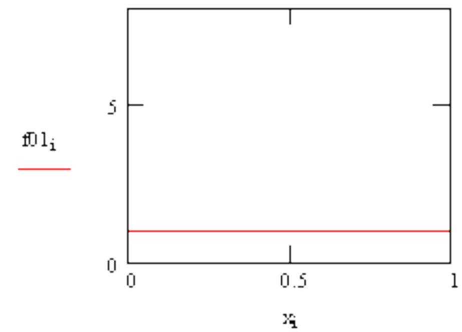
$$p_n(x) = \begin{cases} 1\,, & 0 \le x \le 1 \\ 0\,, & \text{otherwise} \end{cases}$$

An Example of Central Limit Theorem

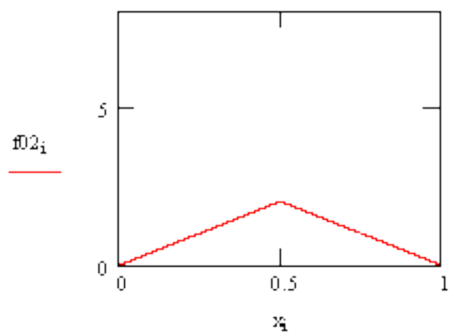


NonNormal Distribution of X

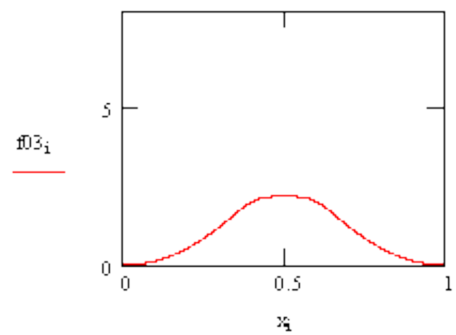
$$\bar{x} = \frac{\frac{\sum x_n}{n}}{n}$$



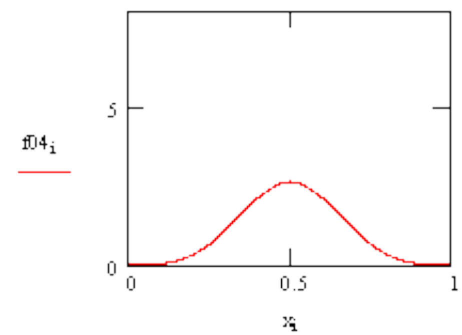
NonNormal Distribution of X



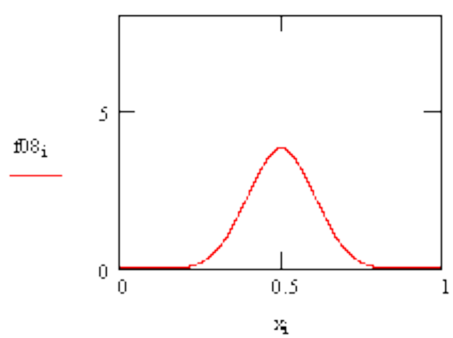
Distribution of Xbar when sample size is 2



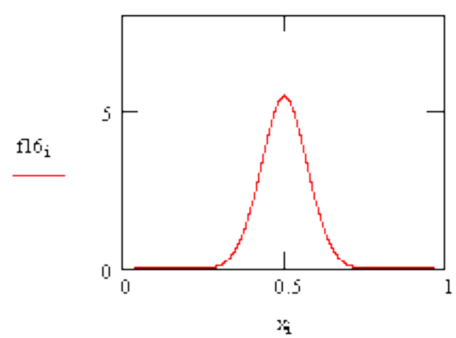
Distribution of Xbar when sample size is 3



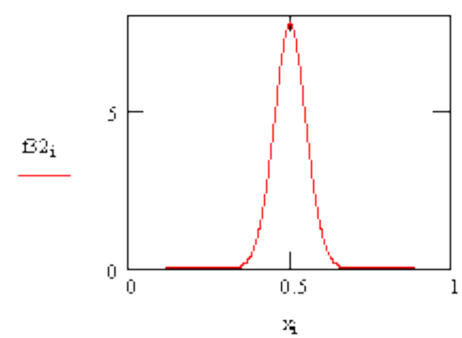
Distribution of Xbar when sample size is 4



Distribution of Xbar when sample size is 8



Distribution of Xbar when sample size is 16



Distribution of Xbar when sample size is 32

Why Gaussian Random Variable is Important?

When a quantity is derived as the result of a large number of accumulative effects, and each individual effect has a small contribution to the final outcome, then the distribution of the quantity tends to follow Gaussian distribution.

Radioactive Disintegration – Bernoulli Process

Consider the radioactive disintegration process in a sample, it follows the following four conditions:

- ☞ It consists of N trials.
- ☞ Each trial has a binary outcome: success or failure (decay or not).
- ☞ The probability of success (decay) is a constant from trial to trial – all atoms have equal probability to decay.
- ☞ The trials are independent.

In statistics, these four conditions characterize a Bernoulli process.

Poisson Process

The counting statistics related to nuclear decay processes is often more conveniently described by the Poisson distribution, is related to situations that involves a collection of multiple trials that satisfy the following conditions:

1. The number of trials, N, is very large, e.g. N>>1.
2. Each trial is independent.
3. The probability that each single trial is successful is a constant and approaching zero, p<<1. So the number of successful trials is fluctuating around a finite number.

Binomial Distribution and Poisson Distribution

Binomial distribution

The probability of observing n successful trails out of a total of N independent trails:

$$P_n = \binom{N}{n} p^n q^{N-n}$$

mean of the observed number of successful trails:

$$\mu \equiv \sum_{n=0}^{N} n \cdot P_n = \sum_{n=0}^{N} n \cdot \binom{N}{n} p^n q^{N-n} = Np$$

Standard deviation:

$$Std(n) \equiv \sqrt{\sum_{n=0}^{N} (n-\mu)^2 \cdot P_n} = \sqrt{Npq}$$

Poisson distribution when N>>1, p<<1

$$P(n \mid \mu) = \frac{\mu^n}{n!} e^{-\mu}$$

Mean of n:

$$\mu(n) = \mu = N \cdot p$$

Standard deviation:

$$\sigma = \sqrt{\mu} = \sqrt{Np}$$

Gaussian distribution, If N is further increased, and p is further decreased

$$p(x \mid \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Error and Error Propagation

Two ways to express the error associated with a given measurement:

Probable error:

- ☞ The symmetric range about the mean, within which there is 50% chance that a measurement will fall.
- ☞ The width of the range depends on the distribution of the variable. For example, for Gaussian distributed error, the probable error is $\pm 0.675\ \sigma$.

Fractional standard deviation:

- ☞ The ratio of the standard deviation and the mean of the distribution of the random variable.
- ☞ For Poisson distributed random variable, the fractional standard deviation is simply

$$\frac{\sigma}{\mu} = \frac{1}{\sqrt{\mu}}$$

Error Propagation

In some situations, the variable of interest (Q) is not measured directly, but derived as a function of more than one independent random variable whose values are directly measured. The error on the measured values is propagated into the uncertainty on the resultant quantity Q.

Suppose a quantity Q(x,y) that depends on two independent random variables x and y.

The sample mean and variance of variables x and y are derived as σ_x and σ_y, by repeating measurements.

The standard deviation of the indirect quantity Q is approximately given by

$$\sigma_Q^{\ 2} \cong \left(\frac{\partial Q}{\partial x}\right)^2 \sigma_x^{\ 2} + \left(\frac{\partial Q}{\partial y}\right)^2 \sigma_y^{\ 2}$$

$$\sigma_Q^{\ 2} \cong \sum_i \left(\frac{\partial Q}{\partial x_i}\right)^2 \sigma_{x_i}^{\ 2}$$

Error and Error Propagation

A Taylor series of a real function of a single variable, f(x), around a point x_0 is given by

$$f(x_0+\Delta x) = f(x_0) + f_x(x_0)\Delta x + \frac{1}{2!} f_{xx}(x_0)(\Delta x)^2 + \frac{1}{3!} f_{xxx}(x_0)(\Delta x)^3 + \ldots\ldots$$

where

$$f_{xx}(x_0) = \left[\frac{d}{dx}\frac{d}{dx} f(x)\right]\Bigg|_{x=x_0}$$

A Taylor series of a real function of two variables, f(x,y), is given by

$$\begin{aligned}
&f(x_0+\Delta x, y_0+\Delta y) = \\
&f(x_0,y_0) + \left[f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y\right] \\
&+\frac{1}{2!}\left[f_{xx}(x_0,y_0)(\Delta x)^2 + 2 f_{xy}(x_0,y_0)\Delta x\Delta y + f_{yy}(x_0,y_0)(\Delta y)^2\right] \\
&+\frac{1}{3!}\left[f_{xxx}(x_0,y_0)(\Delta x)^3 + 3 f_{xxy}(x_0,y_0)(\Delta x)^2(\Delta y) + 3 f_{xyy}(x_0,y_0)(\Delta x)(\Delta y)^2 + f_{yyy}(x_0,y_0)(\Delta y)^3\right] + \ldots\ldots
\end{aligned}$$

Error Propagation

We determine the standard deviation of a quantity $Q(x, y)$ that depends on two independent, random variables x and y. A sample of N measurements of the variables yields pairs of values, x_i and y_i, with $i = 1, 2, \ldots, N$. For the sample one can compute the means, $\bar{x}$ and $\bar{y}$; the standard deviations, σ_x and σ_y; and the values $Q_i = Q(x_i, y_i)$. We assume that the scatter of the x_i and y_i about their means is small. We can then write a power-series expansion for the Q_i about the point $(\bar{x}, \bar{y})$, keeping only the first powers. Thus,

$$Q_i = Q(x_i, y_i) \cong Q(\bar{x}, \bar{y}) + \frac{\partial Q}{\partial x}(x_i - \bar{x}) + \frac{\partial Q}{\partial y}(y_i - \bar{y}), \qquad \text{(E.36)}$$

where the partial derivatives are evaluated at $x = \bar{x}$ and $y = \bar{y}$.

Error Propagation

where the partial derivatives are evaluated at $x = \bar{x}$ and $y = \bar{y}$. The mean value of Q_i is simply

$$\overline{Q} \equiv \frac{1}{N} \sum_{i=1}^{N} Q_i = \frac{1}{N} \sum_{i=1}^{N} Q(\bar{x}, \bar{y}) = \frac{1}{N} NQ(\bar{x}, \bar{y}) = Q(\bar{x}, \bar{y}), \qquad \text{(E.37)}$$

since the sums of the $x_i - \bar{x}$ and $y_i - \bar{y}$ over all i in Eq. (E.36) are zero, by definition of the mean values. Thus, the mean value of Q is the value of the function $Q(x, y)$ calculated at $x = \bar{x}$ and $y = \bar{y}$.

Error and Error Propagation

The variance of the Q_i is given by

$$\sigma_Q^2 = \frac{1}{N}\sum_{i=1}^{N}(Q_i - \overline{Q})^2. \quad \text{(E.38)}$$

$$Q_i = Q(x_i, y_i) \cong Q(\bar{x}, \bar{y}) + \frac{\partial Q}{\partial x}(x_i - \bar{x}) + \frac{\partial Q}{\partial y}(y_i - \bar{y}), \quad \text{(E.36)}$$

Applying Eq. (E.36) with $\overline{Q} = Q(\bar{x}, \bar{y})$, we find that

$$\sigma_Q^2 = \frac{1}{N}\sum_{i=1}^{N}\left[\frac{\partial Q}{\partial x}(x_i - \bar{x}) + \frac{\partial Q}{\partial y}(y_i - \bar{y})\right]^2 \quad \text{(E.39)}$$

$$= \left(\frac{\partial Q}{\partial x}\right)^2 \frac{1}{N}\sum_{i=1}^{N}(x_i - \bar{x})^2 + \left(\frac{\partial Q}{\partial y}\right)^2 \frac{1}{N}\sum_{i=1}^{N}(y_i - \bar{y})^2$$

$$+ 2\left(\frac{\partial Q}{\partial x}\right)\left(\frac{\partial Q}{\partial y}\right)\frac{1}{N}\sum_{i=1}^{N}(x_i - \bar{x})(y_i - \bar{y}). \quad \text{(E.40)}$$

Variance of y, or $\sigma^2(y)$

Covariance, Cov(x, y)

Error and Error Propagation

The last term, called the *covariance* of x and y, vanishes for large N if the values of x and y are uncorrelated. (The factors $y_i - \bar{y}$ and $x_i - \bar{x}$ are then just as likely to be positive as negative, and the covariance also decreases as $1/N$). We are left with the first two terms, involving the variances of the x_i and y_i:

$$\sigma_Q^2 = \left(\frac{\partial Q}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial Q}{\partial y}\right)^2 \sigma_y^2. \qquad \text{(E.41)}$$

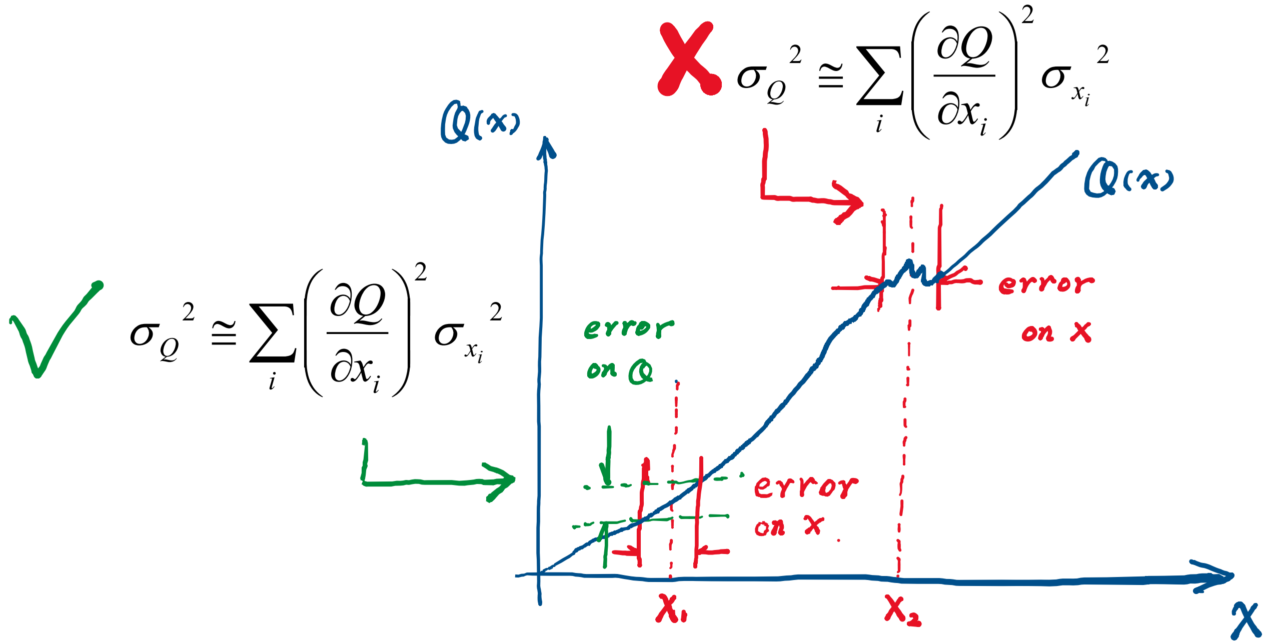
This is one form of the error propagation formula, which is easily generalized to a function Q of any number of independent random variables.

Assumptions ??

Error Propagation Formula

The error propagation formula is exact only when

- the two variables, x and y, are independent to each other,
- and when Q(x,y) could be approximated as a linear function of both x and y.



Error Propagation Formula

The error propagation formula is exact only when

- the two variables, x and y, are independent to each other,
- and when Q(x,y) could be approximated as a linear function of both x and y.

Note that the formula would break down when the second and third and higher order partial derivatives are not negligible.

$$
\begin{aligned}
&f(x_0+\Delta x, y_0+\Delta y) = \\
&f(x_0,y_0)+\left[f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y\right] \\
&+\frac{1}{2!}\left[f_{xx}(x_0,y_0)(\Delta x)^2 + 2f_{xy}(x_0,y_0)\Delta x\Delta y + f_{yy}(x_0,y_0)(\Delta y)^2\right] \\
&+\frac{1}{3!}\left[f_{xxx}(x_0,y_0)(\Delta x)^3 + 3f_{xxy}(x_0,y_0)(\Delta x)^2(\Delta y) + 3f_{xyy}(x_0,y_0)(\Delta x)(\Delta y)^2 + f_{yyy}(x_0,y_0)(\Delta y)^3\right] + \ldots\ldots
\end{aligned}
$$

Error Propagation

Case 1: Sums or differences of counts – u is the sum or difference of two random numbers representing counts measured in two independent experiments.

$$u = x + y \quad \text{or} \quad u = x - y$$

$$\Downarrow$$

$$\sigma_u = \sqrt{\sigma_x^2 + \sigma_y^2}$$

Example: estimating the net counts from a sample.

$$\text{net counts} = \text{total counts} - \text{background counts}$$

or

$$u = x - y$$

Error Propagation

Case 2: Multiplication or division by a constant

$$u = Ax$$

$$\Downarrow$$

$$\sigma_u = A\sigma_x$$

Example: estimating the count rate, $\text{counting rate} \equiv r = \frac{x}{t}$

Assuming that the error in the measuring time is negligible, we get

$$\sigma_r = \frac{\sigma_x}{t}$$

Error Propagation

Case 3: Multiplication or division of counts

$$u = xy, \quad \frac{\partial u}{\partial x} = y \quad \frac{\partial u}{\partial y} = x$$

Using the equation

$$\sigma_Q^{\ 2} \cong \sum_i \left(\frac{\partial Q}{\partial x_i} \right)^2 \sigma_{x_i}^{\ 2}$$

One gets

$$\boxed{\left(\frac{\sigma_u}{u} \right)^2 = \left(\frac{\sigma_x}{x} \right)^2 + \left(\frac{\sigma_y}{y} \right)^2}$$

Error Propagation in Net Count Rate Measurement

As an application of the error-propagation formula, Eq. (11.46), we find the standard deviation of the net count rate of a sample, obtained experimentally as the difference between gross and background count rates, r_g and r_b. As with gross counting, one also measures the number n_b of background counts in a time t_b. The net count rate ascribed to the sample is then the difference

$$r_n = r_g - r_b = \frac{n_g}{t_g} - \frac{n_b}{t_b}. \qquad \sigma_Q^2 = \left(\frac{\partial Q}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial Q}{\partial y}\right)^2 \sigma_y^2. \tag{11.49}$$

To find the standard deviation of r_n, we apply Eq. (11.46) with $Q = r_n$, $x = n_g$, and $y = n_b$. From Eq. (11.49) we have $\partial r_n / \partial n_g = 1/t_g$ and $\partial r_n / \partial n_b = -1/t_b$. Thus, the standard deviation of the net count rate is given by

Assuming no error on t

$$\sigma_{nr} = \sqrt{\frac{\sigma_g^2}{t_g^2} + \frac{\sigma_b^2}{t_b^2}} = \sqrt{\sigma_{gr}^2 + \sigma_{br}^2}. \tag{11.50}$$

where $\sigma_{nr}^2 \equiv \sigma^2(r_n)$, $\sigma_g^2 \equiv \sigma^2(n_g)$, $\sigma_g^2 \equiv \sigma^2(n_b)$,

and $\sigma_{gr}^2 = \sigma^2\left(\frac{n_g}{t_g}\right)$, $\sigma_{br}^2 = \sigma^2\left(\frac{n_b}{t_b}\right)$.

Turner, pp. 324.

Error Propagation in Net Count Rate Measurement

To find the standard deviation of r_n, we apply Eq. (11.46) with $Q = r_n$, $x = n_g$, and $y = n_b$. From Eq. (11.49) we have $\partial r_n / \partial n_g = l/t_g$ and $\partial r_n / \partial n_b = -1/t_b$. Thus, the standard deviation of the net count rate is given by

$$\sigma_{nr} = \sqrt{\frac{\sigma_g^2}{t_g^2} + \frac{\sigma_b^2}{t_b^2}} = \sqrt{\sigma_{gr}^2 + \sigma_{br}^2}. \tag{11.50}$$

Here σ_g and σ_b are the standard deviations of the numbers of gross and background counts, and σ_{gr} and σ_{br} are the standard deviations of the gross and background count rates. Equation (11.50) expresses the well-known result for the standard deviation of the sum or difference of two Poisson or normally distributed random variables. Using n_g and n_b as the best estimates of the means of the gross and background distributions and assuming that the numbers of counts obey Poisson statistics, we have $\sigma_g^2 = n_g$ and $\sigma_b^2 = n_b$. Therefore, the last equation can be written

$$\sigma_{nr} = \sqrt{\frac{n_g}{t_g^2} + \frac{n_b}{t_b^2}} = \sqrt{\frac{r_g}{t_g} + \frac{r_b}{t_b}}, \tag{11.51}$$

Error Propagation in Net Count Rate Measurement

As an application of the error-propagation formula, Eq. (11.46), we find the standard deviation of the net count rate of a sample, obtained experimentally as the difference between gross and background count rates, r_g and r_b. As with gross counting, one also measures the number n_b of background counts in a time t_b. The net count rate ascribed to the sample is then the difference

$$r_n = r_g - r_b = \frac{n_g}{t_g} - \frac{n_b}{t_b}. \qquad \sigma_Q^2 = \left(\frac{\partial Q}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial Q}{\partial y}\right)^2 \sigma_y^2. \tag{11.49}$$

To find the standard deviation of r_n, we apply Eq. (11.46) with $Q = r_n$, $x = n_g$, and $y = n_b$. From Eq. (11.49) we have $\partial r_n/\partial n_g = 1/t_g$ and $\partial r_n/\partial n_b = -1/t_b$. Thus, the standard deviation of the net count rate is given by

Assuming no error on t

$$\sigma_{nr} = \sqrt{\frac{\sigma_g^2}{t_g^2} + \frac{\sigma_b^2}{t_b^2}} = \sqrt{\sigma_{gr}^2 + \sigma_{br}^2}. \tag{11.50}$$

or

$$\sigma_{nr} = \sqrt{\frac{n_g}{t_g^2} + \frac{n_b}{t_b^2}} = \sqrt{\frac{r_g}{t_g} + \frac{r_b}{t_b}},$$

Turner, pp. 324.