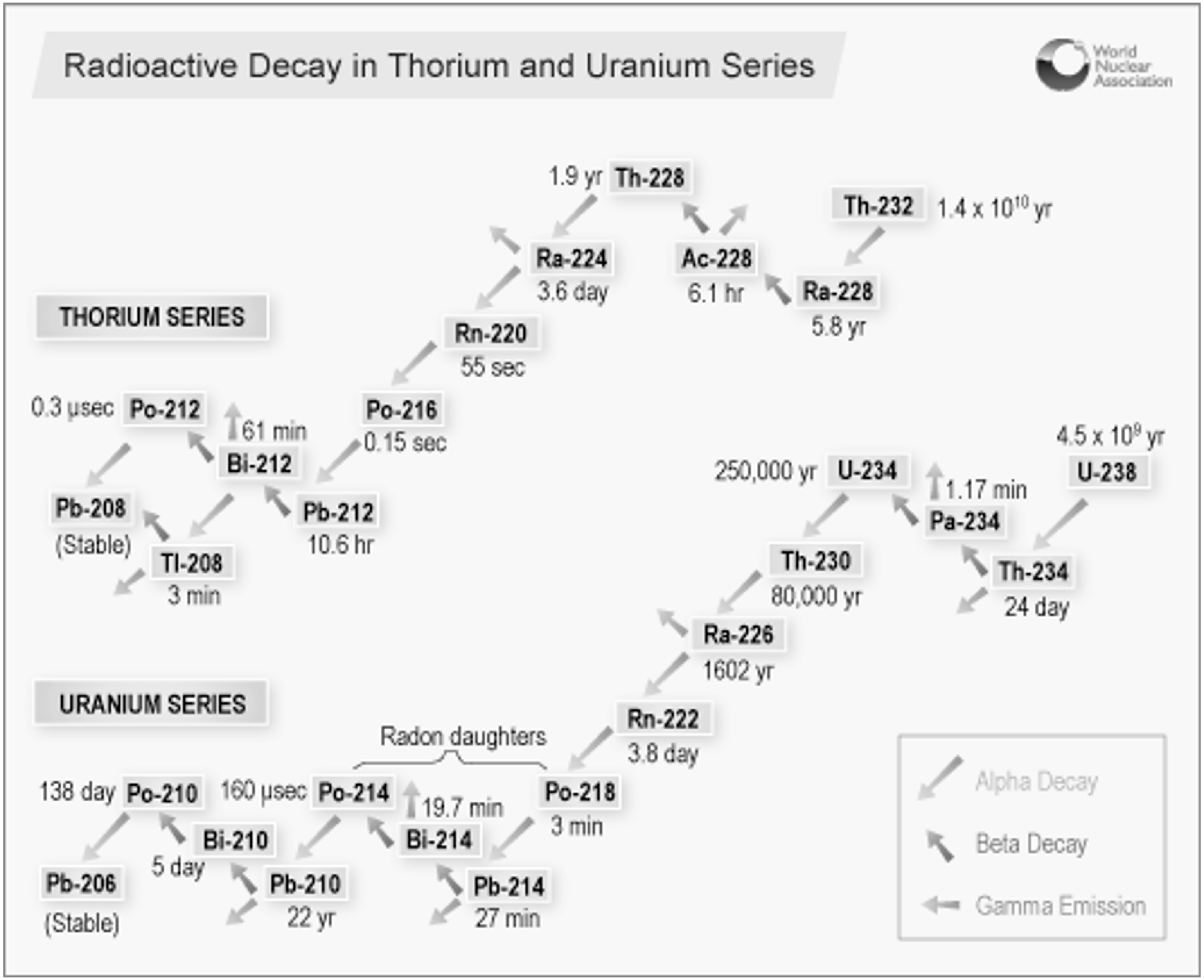
Chapter 3.2: Transformation Kinetics



http://www.world-nuclear.org/info/inf30.html

Serial Transformation

In many situations, the parent nuclides produce one or more radioactive offsprings in a chain. In such cases, it is important to consider the radioactivity from both the parent and the daughter nuclides as a function of time.

$$^{90}_{36}\mathrm{Kr} \xrightarrow[33\ \mathrm{s}]{\beta} {}^{90}_{37}\mathrm{Rb} \xrightarrow[2.74\ \mathrm{min}]{\beta} {}^{90}_{38}\mathrm{Sr} \xrightarrow[28.8\ \mathrm{years}]{\beta} {}^{90}_{39}\mathrm{Y} \xrightarrow[64.2\ \mathrm{h}]{\beta} {}^{90}_{40}\mathrm{Zr}.$$

- Due to their short half lives, ^{90}Kr and ^{90}Rb will be completely transformed, results in a rapid building up of ^{90}Sr.
- ^{90}Y has a much shorter half-life compared to ^{90}Sr. After a certain period of time, the instantaneous amount of ^{90}Sr transformed per unit time will be equal to that of ^{90}Y.
- In this case, ^{90}Y is said to be in a secular equilibrium.

Transformation Kinetics

Exponential Decay

- Different isotopes are characterized by their different rate of transformation (decay).
- The activity of a pure radionuclide decreases exponentially with time. For a given sample, the number of decays within a unit time window around a given time t is a Poisson random variable, whose expectation is given by

$$Q = Q_0 e^{-\lambda \cdot t}$$

- The decay constant λ is the probability of a nucleus of the isotope undergoing a decay within a unit period of time.

Why Exponential Decay?

The activity of a pure radionuclide decreases exponentially with time, as we now show. If N represents the number of atoms of a radionuclide in a sample at any given time, then the change dN in the number during a short time dt is proportional to N and to dt. Letting λ be the constant of proportionality, we write

$$dN = -\lambda N\,dt.$$

The decay rate, A, is given by

$$A = -\frac{dN}{dt} = \lambda N.$$

Separate variables in above equation, we have

$$\frac{dN}{N} = -\lambda\,dt.$$

Integration of both sides gives

$$\ln N = -\lambda t + c,$$

The decay constant λ is the probability of a nucleus of the isotope undergoing a decay within a unit period of time.

Why Exponential Decay? (Continued)

where c is a constant. It can be determined by the boundary condition. For example, we assume that when t=0, $N=N_0$, then we have

$$\ln N = -\lambda t + \ln N_0,$$

Therefore

$$\frac{N}{N_0} = e^{-\lambda t}.$$

Similarly, we can write in terms of radioactivity A as

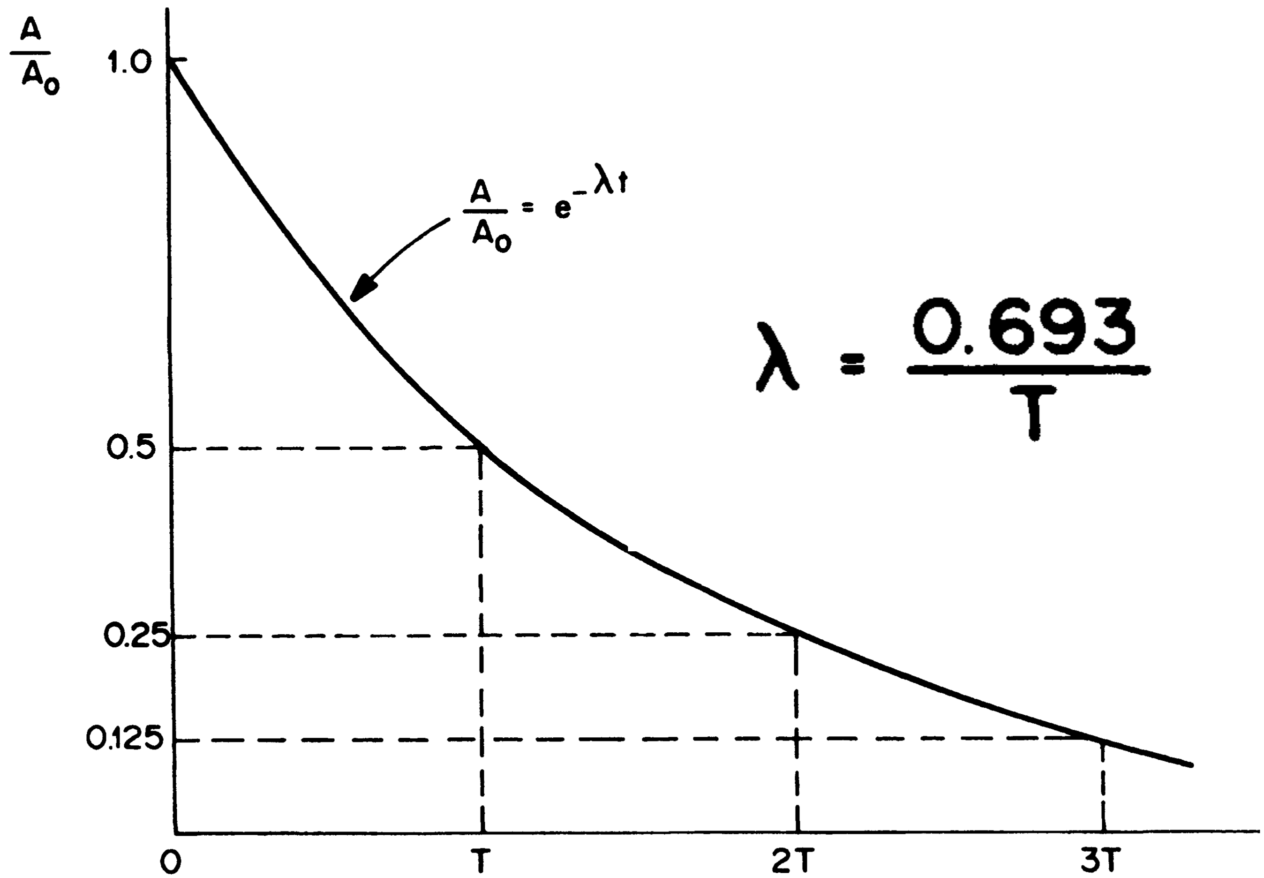
$$\frac{A}{A_0} = e^{-\lambda t}$$

The decay constant λ is the probability of a nucleus of the isotope undergoing a decay within a unit period of time.

Characteristics of Exponential Decay – Half-life

Half-life

The time required for any given radioisotope to decrease to one-half of its original quantity is defined as the half-life, T.



Characteristics of Exponential Decay – half-life

The relationship between half-life T and decay constant λ can be derived by writing

$$\frac{1}{2} = e^{-\lambda T}.$$

Taking the natural logarithm of both sides gives

$$-\lambda T = \ln\left(\tfrac{1}{2}\right) = -\ln 2,$$

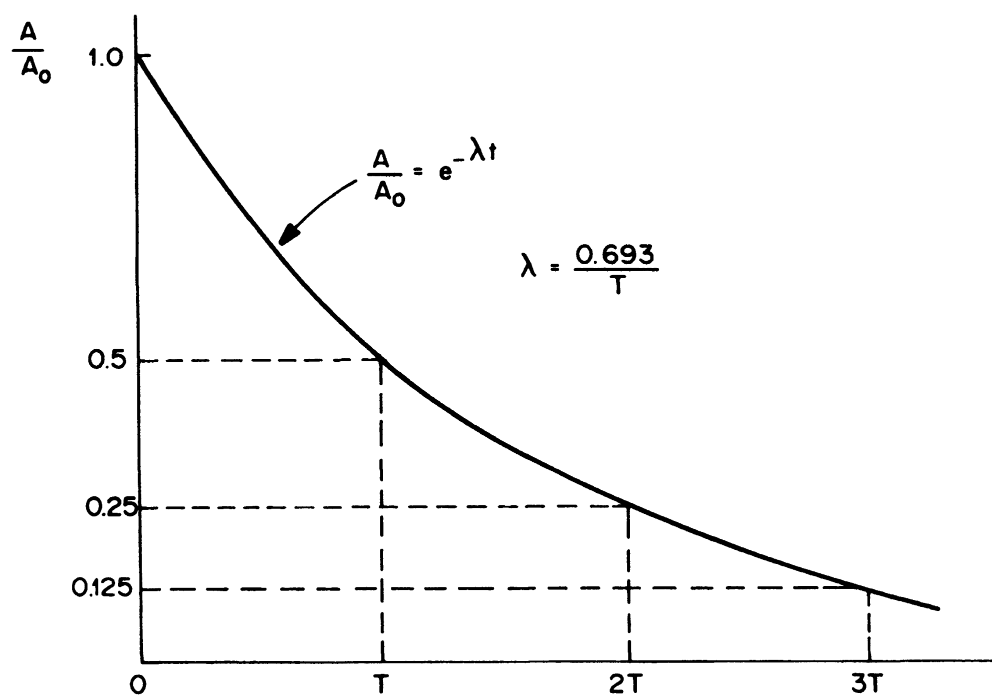
and therefore

$$T = \frac{\ln 2}{\lambda} = \frac{0.693}{\lambda}.$$

Characteristics of Exponential Decay – Average or Mean Life

It is sometimes useful to characterize a radioactive source in terms of the average or mean life of the given isotope, τ. It can be understood as

sum of the lifetimes of the individual atoms divided by the total number of atoms originally present.



$$\tau = \frac{1}{\lambda}$$

Characteristics of Exponential Decay – Average or Mean Life (Continued)

The average or mean life, τ is given by

$$\tau = \int_{t=0}^{\infty} t \cdot \frac{N(t)}{N_0} \cdot \lambda \cdot \mathrm{d}t$$

where N_0 is the number of radioactive atoms in existence at time $t = 0$. Since

$$N = N_0 e^{-\lambda t},$$

we have

$$\tau = \frac{1}{N_0} \int_0^{\infty} t\lambda N_0 e^{-\lambda t}\, \mathrm{d}t.$$

Since

$$\int x e^{ax} dx = \frac{e^{ax}}{a^2}(ax - 1)$$

Characteristics of Exponential Decay – Average or Mean Life (Continued)

we have

$$\tau = \frac{1}{N_0}\int_0^\infty t\lambda N_0 e^{-\lambda t}\,\mathrm{d}t.$$

$$\int x e^{ax} dx = \frac{e^{ax}}{a^2}(ax-1)$$

Therefore

$$\tau = \lambda \frac{e^{-\lambda t}}{\lambda^2}(-\lambda t - 1)\Big|_0^\infty = \frac{1}{\lambda}$$

Units for Radioactivity

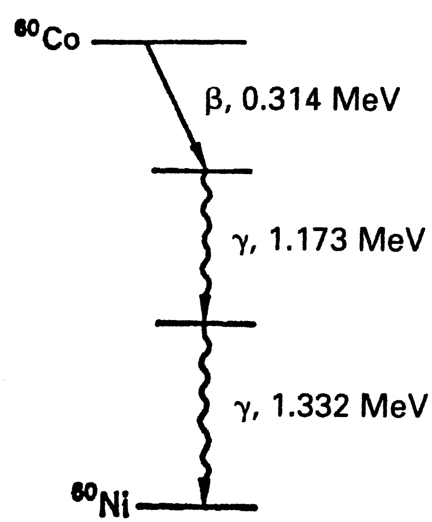
The Becquerel (Bq) – SI standard unit for radioactivity

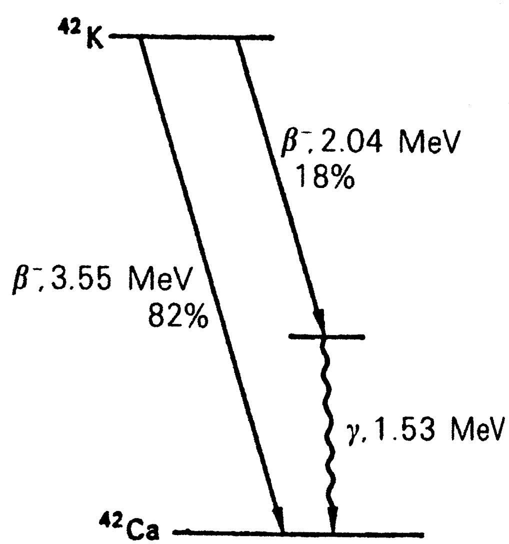
The Becquerel is the quantity of radioactive material in which one atom is transform per second.

$$1Bq = 1tps$$

$$1Curie(Ci) = 3.7 \times 10^{10}\,Bq$$

Note that a Becquerel is not the number of particles emitted by the radioactive isotope in 1 s.





Specific Activity (SA)

Specific activity of a sample is defined as its activity per unit mass, given in units of Bq/g or Ci/g.

Specific activity for pure radioisotopes is defined as the number of Becquerels per unit mass.

$$\text{Specific Activity} = \frac{6.03\times10^{23}\left(atoms/mole\right)}{A\left(g/mole\right)} \times \lambda \quad Bq/g$$

Activity per unit mass

Number of atoms per unit mass

The probably of an atom decaying within a unit time span

SA can be related to the half-life (T) of the radionuclide by

$$SA = \frac{4.18\times10^{23}}{A\cdot T}\, Bq/g$$

Specific Activity (Continued)

An example:

A solution of $Hg(NO_3)_2$ tagged with ^{203}Hg has a specific activity of 1.5×10^5 Bq/mL ($4\ \frac{\mu Ci}{mL}$). If the concentration of mercury in the solution is $5\ \frac{mg}{mL}$,

(a) what is the specific activity of the mercury?

(b) what fraction of the mercury in the $Hg(NO_3)_2$ is ^{203}Hg?

(c) what is the specific activity of the $Hg(NO_3)_2$?

Specific Activity (Continued)

Solution:

(a) The specific activity of the mercury is

$$SA\,(\mathrm{Hg}) = \frac{1.5 \times 10^5\ \mathrm{Bq/mL}}{5\ \mathrm{mg\ Hg/mL}} = 0.3 \times 10^5\ \frac{\mathrm{Bq}}{\mathrm{mg}}\ \mathrm{Hg}.$$

and the specific activity of ^{203}Hg is calculated from

$$SA = \frac{4.18\times10^{23}}{A\cdot T} Bq/g = \frac{4.18\times10^{23}}{203\cdot 46.5d\cdot 24h/d\cdot 3600s/h} Bq/g = 5.2\times10^{14} Bq/g$$

Specific Activity (Continued)

(b) what fraction of the mercury in the $Hg(NO_3)_2$ is ^{203}Hg?

Solution:

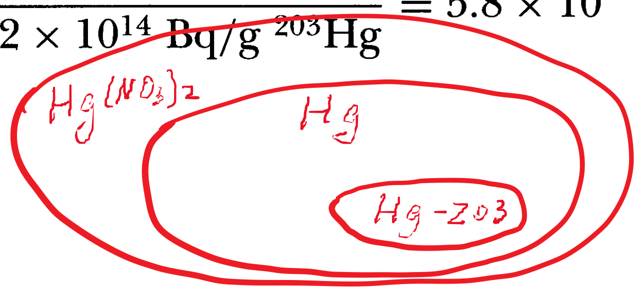
(b) The weight-fraction of mercury that is tagged is given by $\dfrac{SA\,(\mathrm{Hg})}{SA\,(^{203}\mathrm{Hg})}$,

and the specific activity of ^{203}Hg is calculated from

$$SA = \frac{4.18\times10^{23}}{A\cdot T}\,Bq/g = \frac{4.18\times10^{23}}{203\cdot 46.5d\cdot 24h/d\cdot 3600s/h}\,Bq/g = 5.2\times10^{14}\,Bq/g$$

The weight fraction of ^{203}Hg, therefore, is

$$\frac{SA\,(\mathrm{Hg})}{SA\,(^{203}\mathrm{Hg})} = \frac{0.3\times10^{8}\ \mathrm{Bq/g\ Hg}}{5.2\times10^{14}\ \mathrm{Bq/g\ ^{203}Hg}} = 5.8\times10^{-8}\ \frac{\mathrm{g\ ^{203}Hg}}{\mathrm{g\ Hg}}.$$



Specific Activity (Continued)

(c) what is the specific activity of the $Hg(NO_3)_2$?

A solution of $Hg(NO_3)_2$ tagged with ^{203}Hg has a specific activity of 1.5×10^5 Bq/mL $(4\ \frac{\mu Ci}{mL})$. If the concentration of mercury in the solution is $5\ \frac{mg}{mL}$,

Solution:

(c) Since an infinitesimally small fraction of the mercury is tagged with ^{203}Hg, it may be assumed that the formula weight of the tagged Hg $(NO_3)_2$ is 324.63 and that the concentration of Hg $(NO_3)_2$ is

$$\frac{324.63 \text{ mg Hg } (NO_3)_2}{200.61 \text{ mg Hg}} \times \frac{5 \text{ mg Hg}}{\text{mL}} = 8.1 \frac{\text{mg Hg } (NO_3)_2}{\text{mL}}.$$

The specific activity,

$$\frac{1.5 \times 10^5 \text{ Bq/mL}}{8.1 \text{ mg Hg } (NO_3)_2/\text{mL}} = 1.9 \times 10^4 \frac{\text{Bq}}{\text{mg}} \text{ Hg } (NO_3)_2 \left[0.5 \frac{\mu\text{Ci}}{\text{mg}} \text{ Hg } (NO_3)_2\right].$$

Serial Transformation

In many situations, the parent nuclides produce one or more radioactive offsprings in a chain. In such cases, it is important to consider the radioactivity from both the parent and the daughter nuclides as a function of time.

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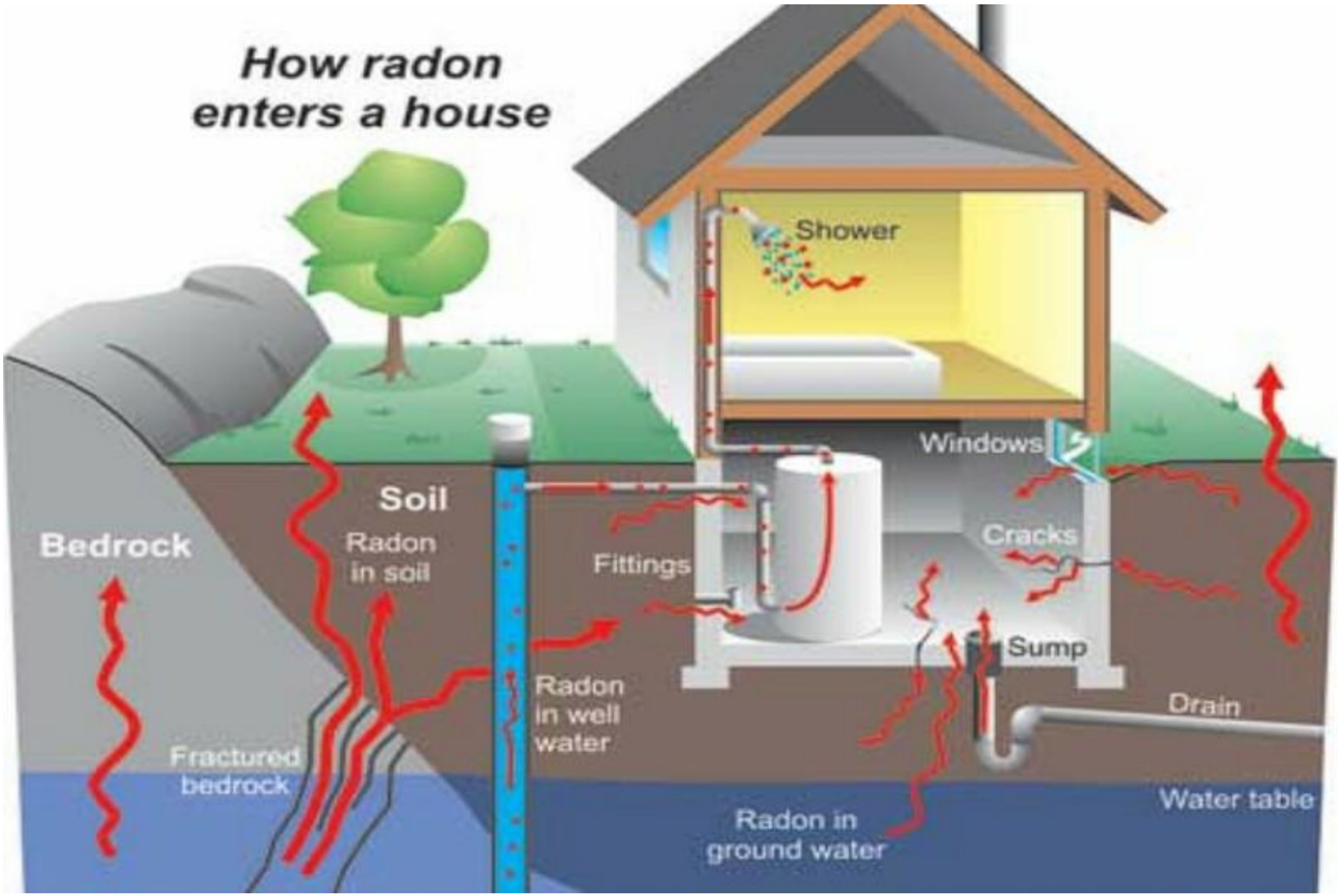
Indoor Radon



Naturally Occurring Radioactivity – Health Concerns of Radon Gas

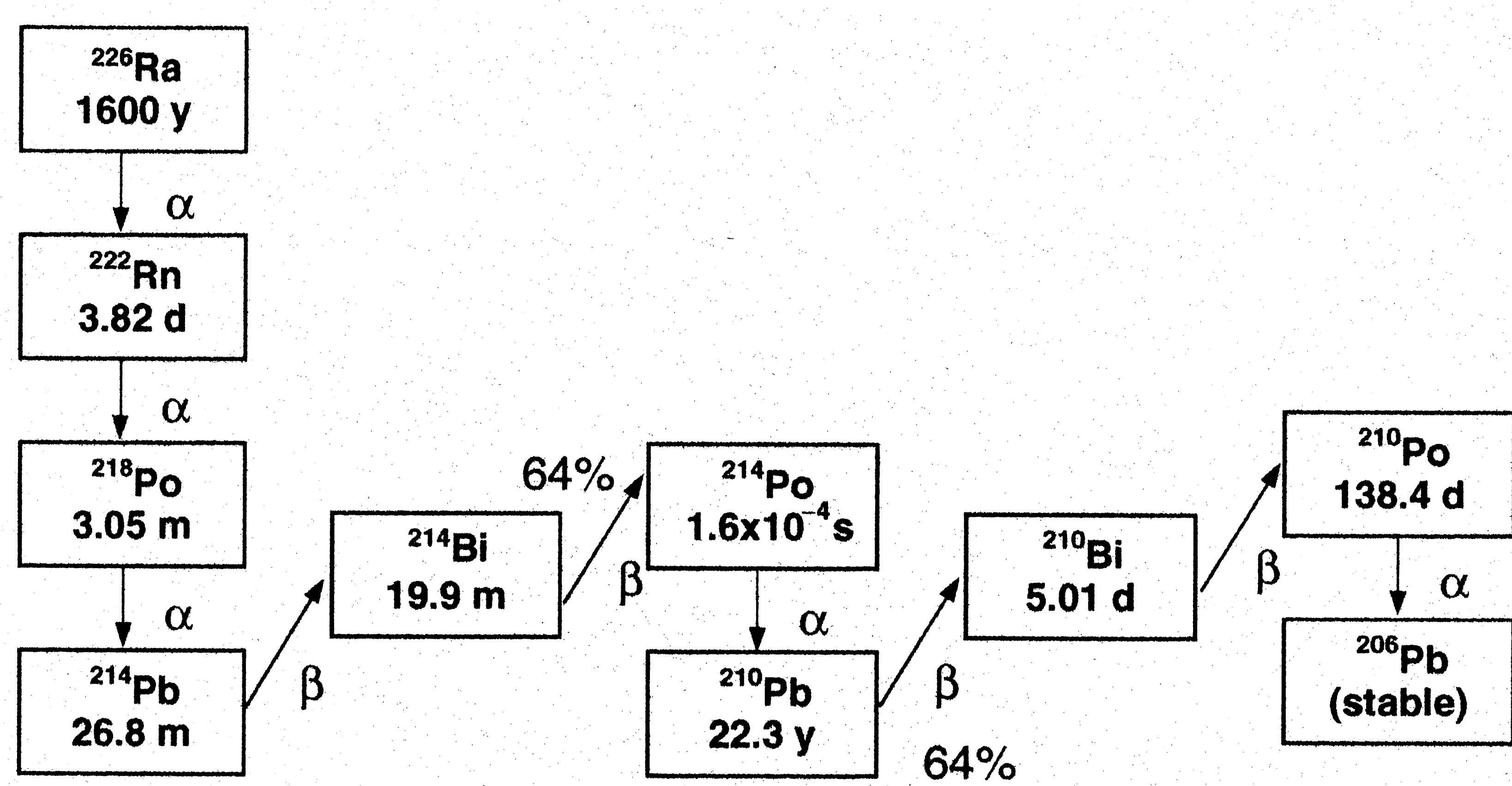
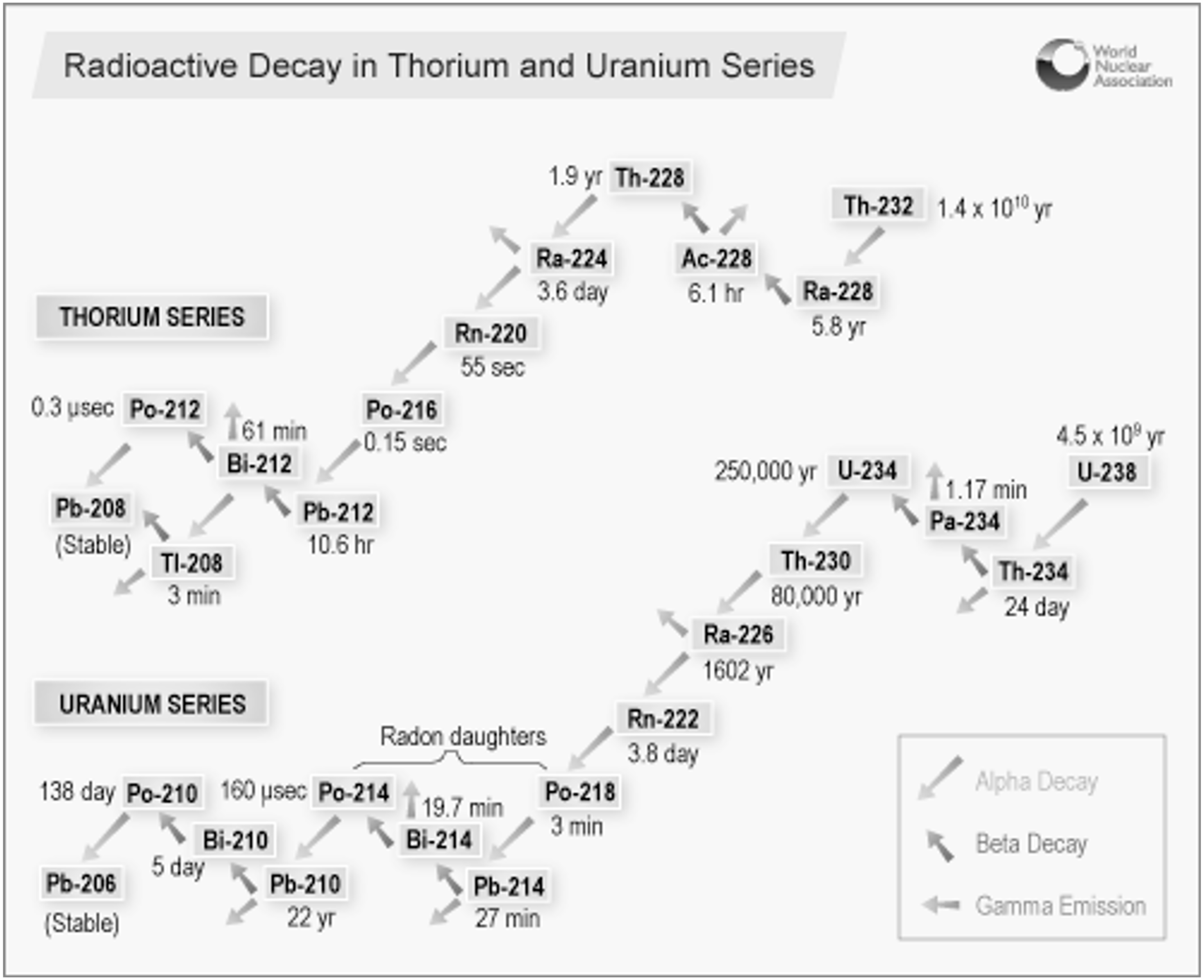


Figure 3.11 The ^{226}Ra decay series.



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General Case

Consider a more general case, in which (a) the half-life of the parent can be of any conceivable value and (b) no restrictions are applied on the relative half-lives of both the parent and the daughter.

$$A \xrightarrow{\lambda_A} B \xrightarrow{\lambda_B} C,$$

The number of atoms of the parent A and the daughter B at any given time t are therefore related by

$$N_B = \frac{\lambda_A N_{A_0}}{\lambda_B - \lambda_A}\left(e^{-\lambda_A t} - e^{-\lambda_B t}\right)$$

Proof of the Previous Serial Decay Equation
From Cember, p123-124

of the daughter, it follows that secular equilibrium is a special case of a more general situation in which the half-life of the parent may be of any conceivable magnitude, but greater than that of the daughter. For this general case, where the parent activity is not relatively constant,

$$A \xrightarrow{\lambda_A} B \xrightarrow{\lambda_B} C,$$

the time rate of change of the number of atoms of species B is given by the differential equation

$$\frac{dN_B}{dt} = \lambda_A N_A - \lambda_B N_B. \tag{4.42}$$

In this equation, $\lambda_A N_A$ is the rate of transformation of species A and is exactly equal to the rate of formation of species B, the rate of transformation of isotope B is $\lambda_B N_B$, and the difference between these two rates at any time is the instantaneous rate of growth of species B at that time.

According to Eq. (4.18), the value of λ_A in Eq. (4.42) may be written as

$$N_A = N_{A_0} e^{-\lambda t}. \tag{4.43}$$

Equation (4.42) may be rewritten, after substituting the expression above for N_A and transposing $\lambda_B N_B$, as

$$\frac{dN_B}{dt} + \lambda_B N_B = \lambda_A N_{A_0} e^{-\lambda_A t}. \tag{4.44}$$

Proof of The Serial Decay Equation (Continued)

y

$$\frac{dN_B}{dt} + \lambda_B N_B = \lambda_A N_{A_0} e^{-\lambda_A t}. \quad \textbf{(4.44)}$$

x P(x) Q(x)

Equation (4.44) is a first-order linear differential equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x), \quad \textbf{(4.45)}$$

and may be integrable by multiplying both sides of the equation by

$$e^{\int P\,dx} = e^{\int \lambda_B\,dt} = e^{\lambda_B t},$$

and the solution to Eq. (4.45) is

$$y e^{\int P\,dx} = \int e^{\int P\,dx} \cdot Q\,dx. \quad \textbf{(4.46)}$$

$$y = \frac{\int e^{\int P(x)\cdot dx} \cdot Q(x) \cdot dx + C}{e^{\int P(x)\,dx}}$$

Since N_B, λ_B, and $\lambda_A N_{A_0} e^{-\lambda_A t}$ from Eq. (4.44) are represented in Eq. (4.46) by y, P, and Q, respectively, the solution of Eq. (4.44) is

$$N_B e^{\lambda_B t} = \int e^{\lambda_B t} \lambda_A N_{A_0} e^{-\lambda_A t}\,dt + C \quad \textbf{(4.47)}$$

or, if the two exponentials are combined, we have

$$N_B e^{\lambda_B t} = \int \lambda_A N_{A_0} e^{(\lambda_B - \lambda_A)t}\,dt + C. \quad \textbf{(4.48)}$$

Proof of The Serial Decay Equation (Continued)

$$N_B e^{\lambda_B t} = \int \lambda_A N_{A_0} e^{(\lambda_B - \lambda_A)t}\, \mathrm{d}t + C. \tag{4.48}$$

If the integrand in Eq. (4.48) is multiplied by the integrating factor $\lambda_B - \lambda_A$, then Eq. (4.48) is in the form

$$\int e^v \mathrm{d}v = e^v + C \tag{4.49}$$

and the solution is

$$N_B e^{\lambda_B t} = \frac{1}{\lambda_B - \lambda_A} \lambda_A N_{A_0} e^{(\lambda_B - \lambda_A)t} + C. \tag{4.50}$$

If $t = 0, N_B = 0$, then

$$N_B = \frac{\lambda_A N_{A0}}{\lambda_B - \lambda_A}\left(\mathrm{e}^{-\lambda_A t} - \mathrm{e}^{-\lambda_B t}\right)$$

General Case

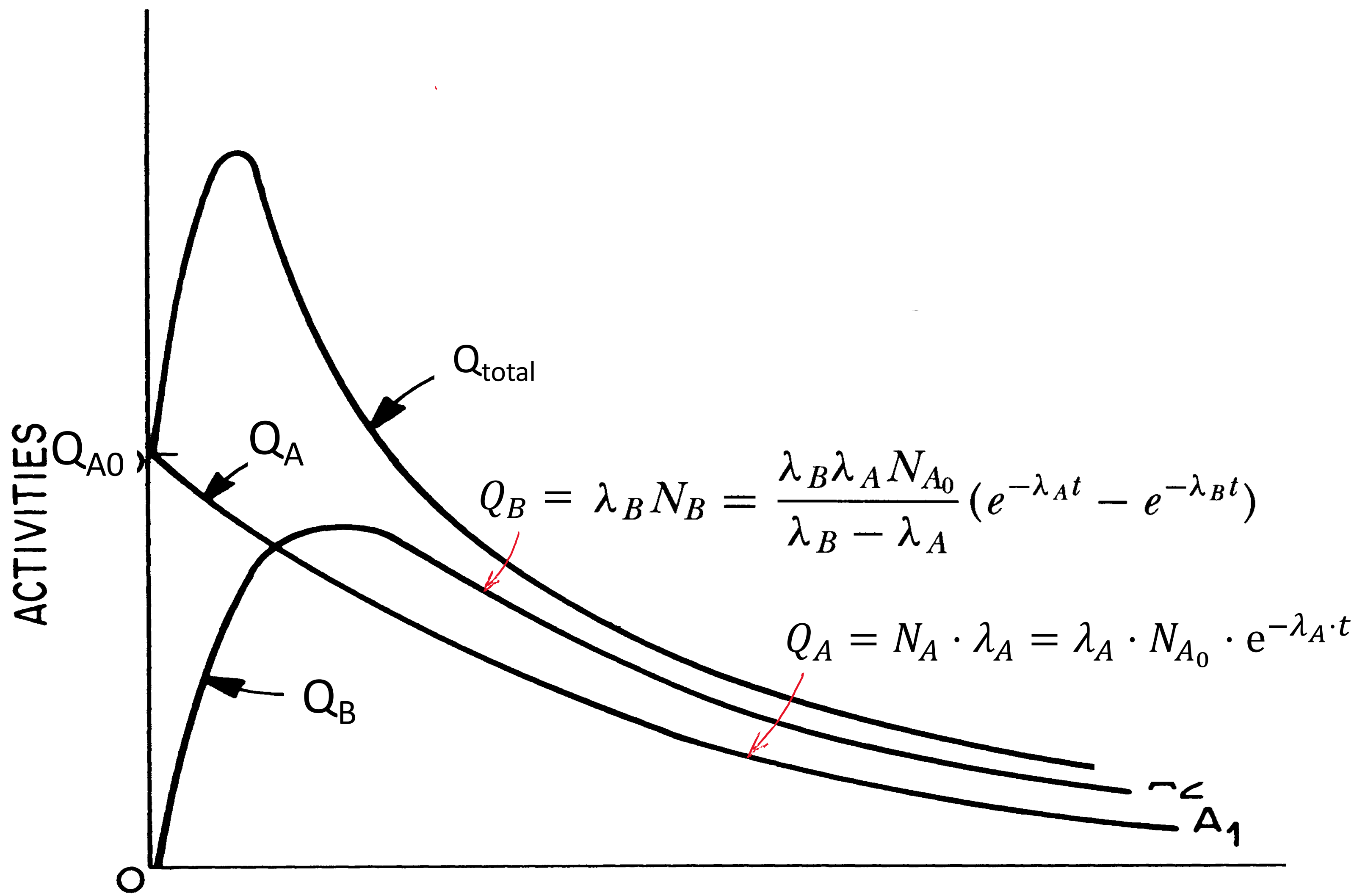
Consider a more general case, in which (a) the half-life of the parent can be of any conceivable value and (b) no restrictions are applied on the relative half-lives of both the parent and the daughter.

$$A \xrightarrow{\lambda_A} B \xrightarrow{\lambda_B} C,$$

The number of atoms of the parent A and the daughter B at any given time t are therefore related by

$$N_B = \frac{\lambda_A N_{A_0}}{\lambda_B - \lambda_A}\left(e^{-\lambda_A t} - e^{-\lambda_B t}\right)$$

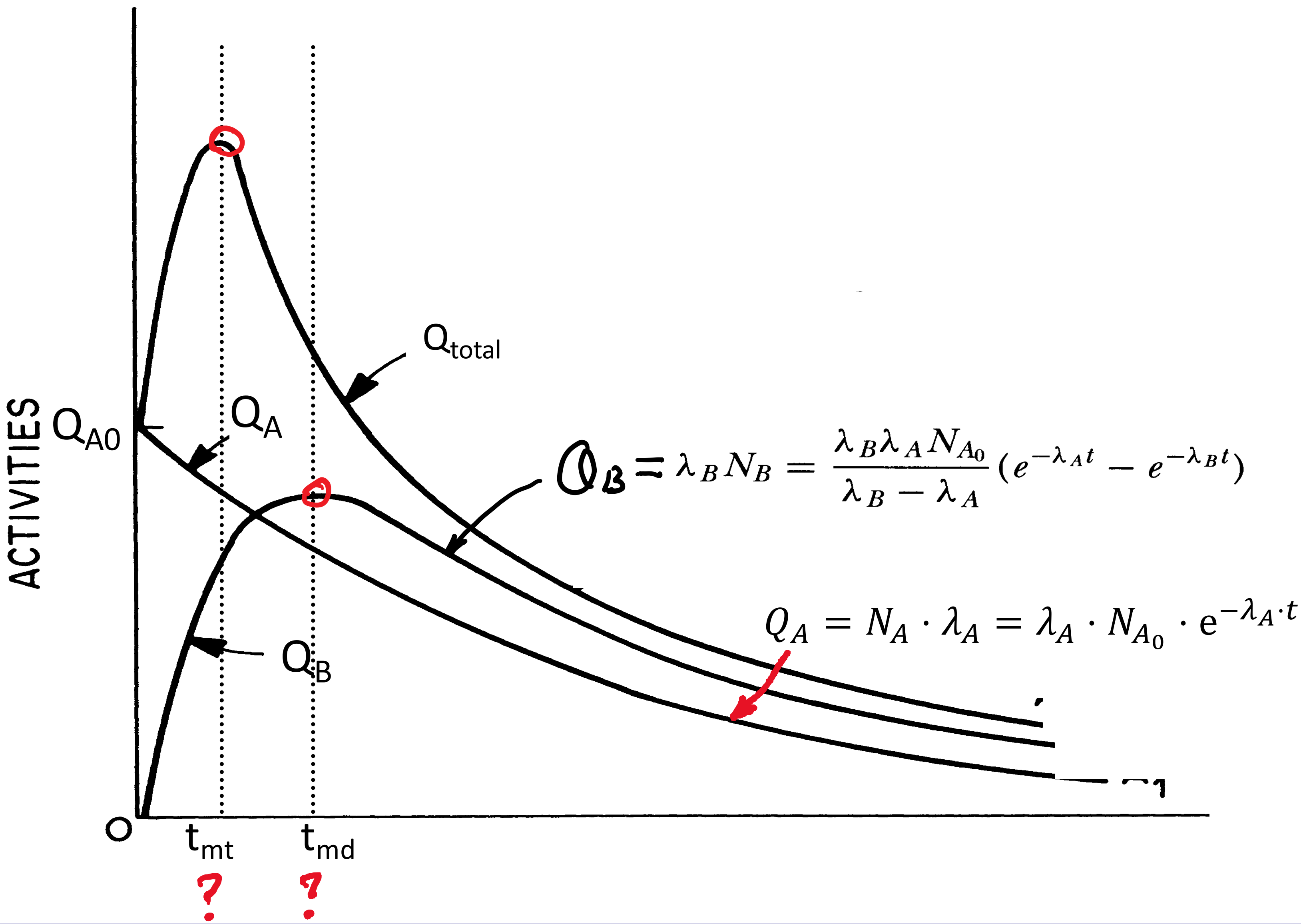
Activities from the Parent and the Daughter



$$Q_B = \lambda_B N_B = \frac{\lambda_B \lambda_A N_{A_0}}{\lambda_B - \lambda_A}\left(e^{-\lambda_A t} - e^{-\lambda_B t}\right)$$

$$Q_A = N_A \cdot \lambda_A = \lambda_A \cdot N_{A_0} \cdot \mathrm{e}^{-\lambda_A \cdot t}$$

Activity Peaking Times Under General Case



Activity Peaking Time Under General Case

The peak-reaching-time for the activity from the daughter can be derived as the following:

Start from the equation for the general case

$$\lambda_B N_B = \frac{\lambda_B \lambda_A N_{A_0}}{\lambda_B - \lambda_A}(e^{-\lambda_A t} - e^{-\lambda_B t})$$

Differentiate respect to t and set to zero

$$\frac{d(\lambda_B N_B)}{dt} = \frac{\lambda_B \lambda_A N_{A_0}}{\lambda_B - \lambda_A}(-\lambda_A e^{-\lambda_A t} + \lambda_B e^{-\lambda_B t}) = 0,$$

$$\lambda_A e^{-\lambda_A t} = \lambda_B e^{-\lambda_B t}$$

and therefore

$$\ln \frac{\lambda_B}{\lambda_A} = (\lambda_B - \lambda_A) t$$

$$t = t_{md} = \frac{\ln(\lambda_B/\lambda_A)}{\lambda_B - \lambda_A} = \frac{2.3 \log(\lambda_B/\lambda_A)}{\lambda_B - \lambda_A}.$$

Activity Peaking Time Under General Case

Similarly, the peak reaching times for the total activity is …

The total activity is $A(t) = \lambda_A N_A + \lambda_B N_B.$

Since

$$A_B(t) = \lambda_B N_B = \frac{\lambda_B \lambda_A N_{A_0}}{\lambda_B - \lambda_A}(e^{-\lambda_A t} - e^{-\lambda_B t}), \quad A_A(t) = N_{A_0} \cdot e^{-\lambda_A \cdot t}$$

then

$$A(t) = \lambda_A N_{A_0} e^{-\lambda_A t} + \frac{\lambda_B \lambda_A N_{A_0}}{\lambda_B - \lambda_A}(e^{-\lambda_A t} - e^{-\lambda_B t}).$$

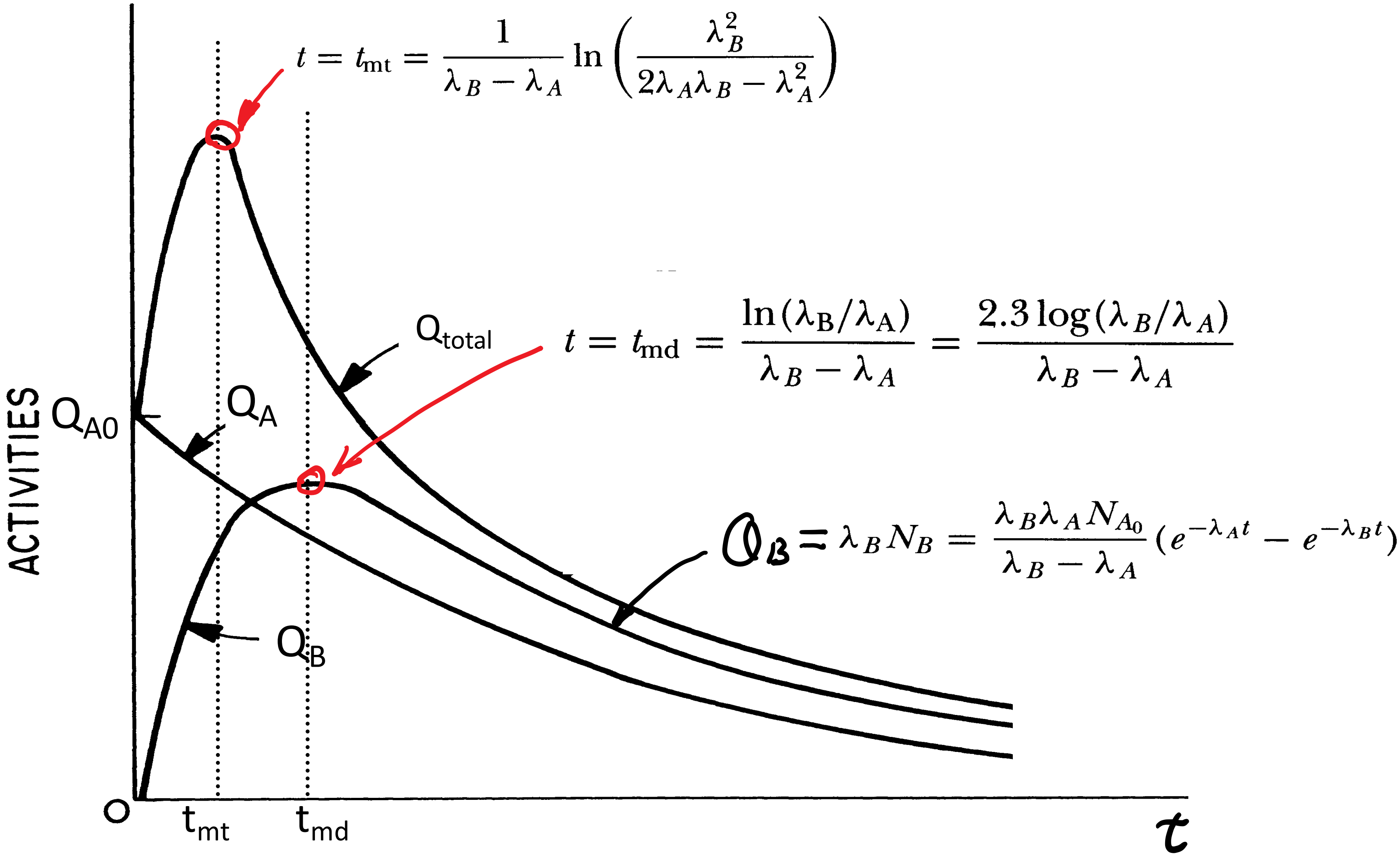
Differentiate respect to t and set to zero, we have

$$\frac{dA(t)}{dt} = -\lambda_A^2 N_{A_0} e^{-\lambda_A t} + \frac{\lambda_B \lambda_A N_{A_0}}{\lambda_B - \lambda_A}(-\lambda_A e^{-\lambda_A t} + \lambda_B e^{-\lambda_B t}) = 0.$$

Solving for t, it can be shown that

$$t = t_{mt} = \frac{1}{\lambda_B - \lambda_A} \ln\left(\frac{\lambda_B^2}{2\lambda_A \lambda_B - \lambda_A^2}\right)$$

Activity Peaking Times Under General Case



Further Discussions on Serial Transformations

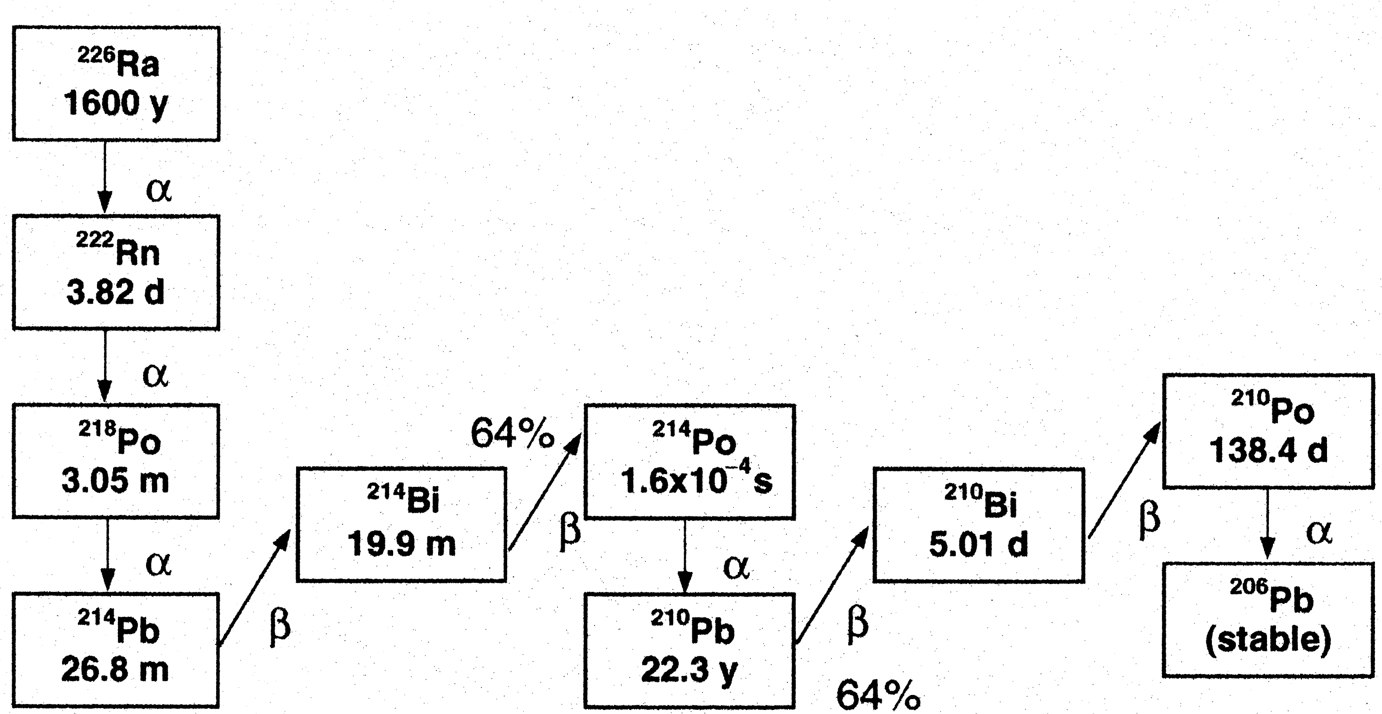


Figure 3.11 The ^{226}Ra decay series.

Further Discussions on Serial Transformations

Now, I know the question burning in your mind is, "What if species C is also radioactive?" This is certainly possible; in fact some of the most important and interesting problems in health physics involve long chains of products, one decaying to the next until a stable species is reached. So now let's solve for the situation:

$$A \xrightarrow{\lambda_A} B \xrightarrow{\lambda_B} C \xrightarrow{\lambda_C} D$$

Further Discussions on Serial Transformations

$$A \xrightarrow{\lambda_A} B \xrightarrow{\lambda_B} C \xrightarrow{\lambda_C} D$$

$$\frac{dN_C}{dt} = \lambda_B N_B - \lambda_C N_C$$

$$\lambda_B N_B = \frac{\lambda_B \lambda_A N_{A_0}}{\lambda_B - \lambda_A}(e^{-\lambda_A t} - e^{-\lambda_B t})$$

The solutions for A and B do not change; they are not dependent on what happens to other members of the chain. The solution for C will be of the form:

$$N_C = N_{A_0}\left(F_1 e^{-\lambda_A T} + F_2 e^{-\lambda_B T} + F_3 e^{-\lambda_C T}\right)$$

where F1, F2, and F3 are coefficients that will depend on the initial conditions of the problem. If we assume that all branching ratios are 1.0 and that:

$N_A(0) = N_{A_0}$, $N_B(0) = 0$, $N_C(0) = 0$ we find:

$$F_1 = \frac{\lambda_A}{\lambda_C - \lambda_A}\frac{\lambda_B}{\lambda_B - \lambda_A}$$

$$F_2 = \frac{\lambda_A}{\lambda_A - \lambda_B}\frac{\lambda_B}{\lambda_C - \lambda_B}$$

$$F_3 = \frac{\lambda_A}{\lambda_A - \lambda_C}\frac{\lambda_B}{\lambda_B - \lambda_C}$$

Activity Peaking Times Under General Case

$$A \xrightarrow{\lambda_A} B \xrightarrow{\lambda_B} C \xrightarrow{\lambda_C} D$$

$$\frac{dN_C}{dt} = \lambda_B N_B - \lambda_C N_C$$

$$\lambda_B N_B = \frac{\lambda_B \lambda_A N_{A_0}}{\lambda_B - \lambda_A}(e^{-\lambda_A t} - e^{-\lambda_B t})$$

The number of atoms of C can be found by substitution into the above equation. The activity of C at any time is found as

$$A_C = A_{A_0}\left(\frac{\lambda_B}{\lambda_B - \lambda_A}\frac{\lambda_C}{\lambda_C - \lambda_A}e^{-\lambda_A t} + \frac{\lambda_B}{\lambda_A - \lambda_B}\frac{\lambda_C}{\lambda_C - \lambda_B}e^{-\lambda_B t} + \frac{\lambda_B}{\lambda_B - \lambda_C}\frac{\lambda_C}{\lambda_A - \lambda_C}e^{-\lambda_C t}\right)$$

Several Special Cases

Secular Equilibrium: $T_A >> T_B$ ($\lambda_A << \lambda_B$) and $t > 7T_B$

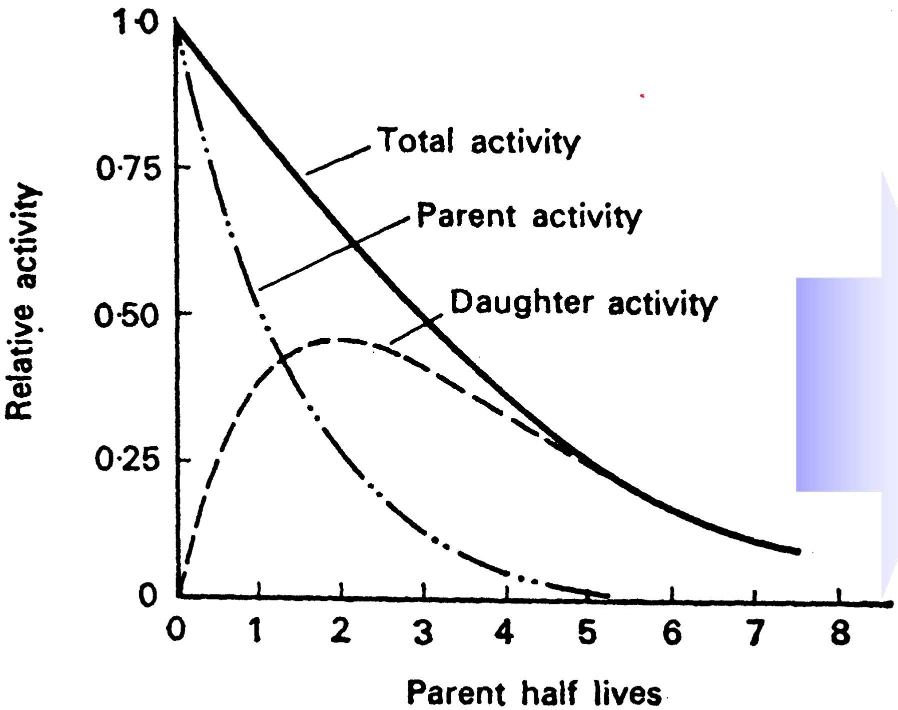
For the following serial transformation:

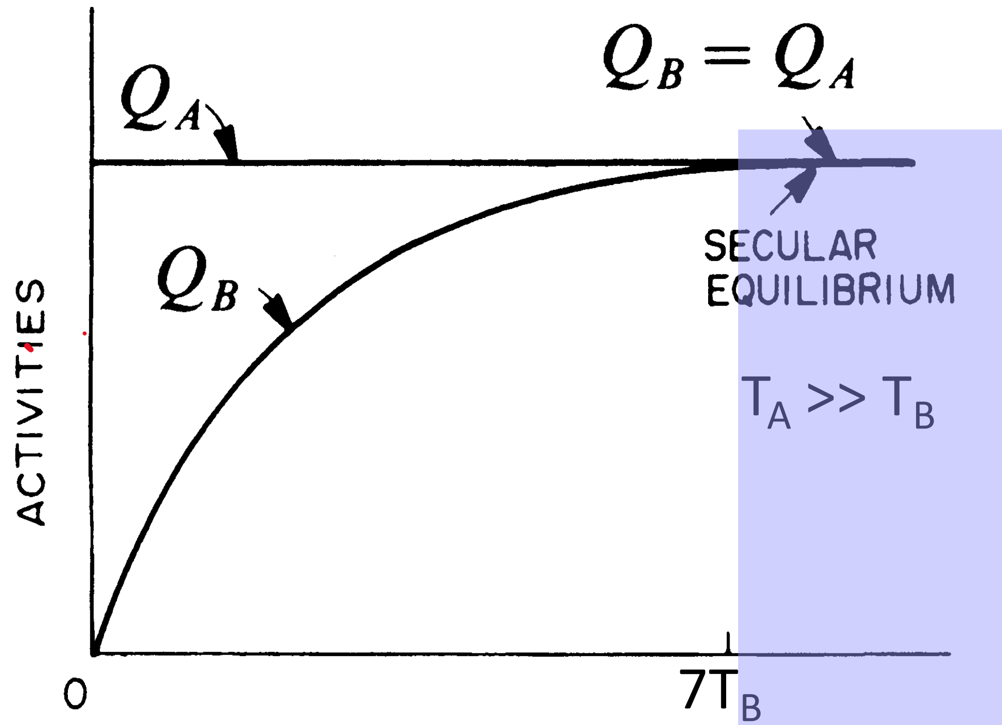
$$A \xrightarrow{\lambda_A} B \xrightarrow{\lambda_B} C,$$

where $\lambda_A << \lambda_B$ and $T_A >> T_B$, B is said to be in secular equilibrium. For example

$$^{90}_{36}\mathrm{Kr} \xrightarrow[33\ \mathrm{s}]{\beta} {}^{90}_{37}\mathrm{Rb} \xrightarrow[2.74\ \mathrm{min}]{\beta} {}^{90}_{38}\mathrm{Sr} \xrightarrow[28.8\ \mathrm{years}]{\beta} {}^{90}_{39}\mathrm{Y} \xrightarrow[64.2\ \mathrm{h}]{\beta} {}^{90}_{40}\mathrm{Zr}.$$

Secular Equilibrium: $T_A >> T_B$ and $t > 7T_B$





General Case

Secular Equilibrium

$$N_B = \frac{\lambda_A N_{A_0}}{\lambda_B - \lambda_A}\left(e^{-\lambda_A t} - e^{-\lambda_B t}\right)$$

$$N_B \approx \frac{\lambda_A N_A}{\lambda_B}\left(1 - \mathrm{e}^{-\lambda_B t}\right)$$

$$Q_B \approx Q_A \cdot \left(1 - \mathrm{e}^{-\lambda_B \cdot t}\right)$$

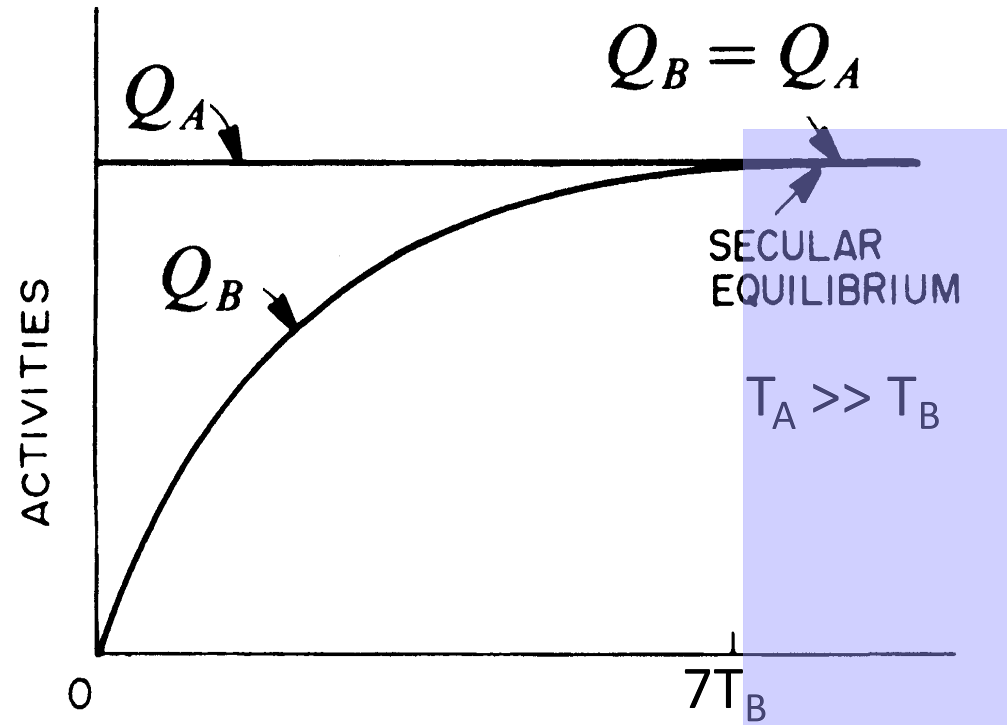
Secular Equilibrium: $T_A >> T_B$ and $t > 7T_B$

From this relationship,

$$N_B \approx \frac{\lambda_A N_A}{\lambda_B}\left(1 - e^{-\lambda_B t}\right)$$

one can see that

1. As the time goes by, $e^{-\lambda}$ decreases and Q_B approaches Q_A At equilibrium, we have



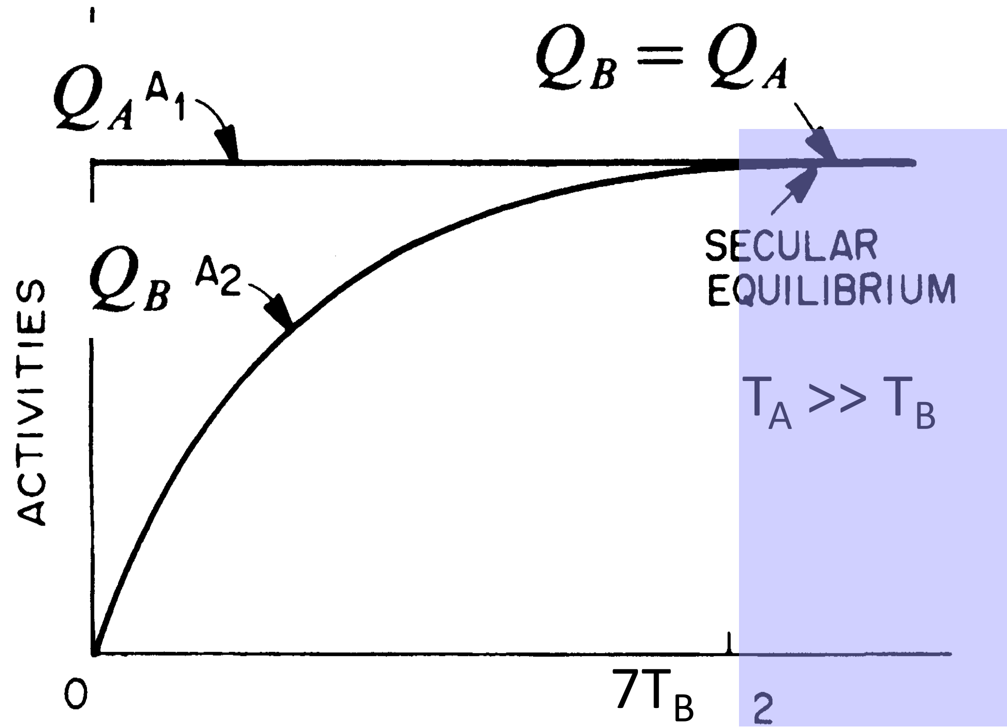
$$\lambda_A N_A = \lambda_B N_B \quad and \quad Q_A = Q_B$$

2. Since A has a relatively long half life, Q_A may be considered as a constant. So the total activity converges to a constant.

Secular Equilibrium: $T_A >> T_B$ and $t > 7T_B$

From this relationship,

$$Q_B = Q_A(1 - e^{-\lambda_B t}),$$



One could also see that

1. As the time goes by, $e^{-\lambda_B t}$ decreases and Q_B approaches Q_A. At equilibrium, we have

$$\lambda_A N_A = \lambda_B N_B \quad and \quad Q_A = Q_B$$

2. Since A has a relatively long half life, Q_A may be considered as a constant. So the total activity converges to a constant.

Activity Peaking Times Under General Case

From <<Radiation Protection and Dosimetry>>, by Michael Stabin.

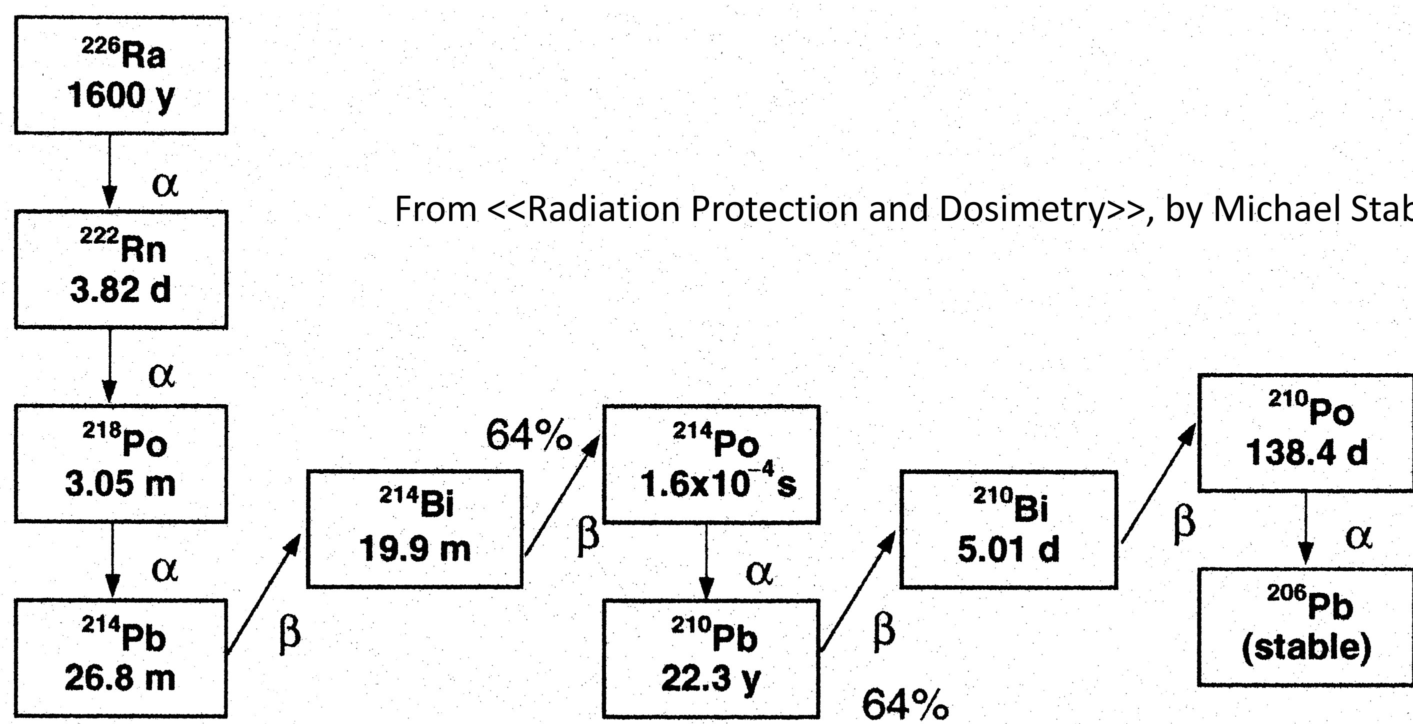


Figure 3.11 The ^{226}Ra decay series.

We can continue on with a species D, E, F, and so on, but the relationships among the species obviously become more complicated and are difficult to categorize. If Species A is very long-lived, however, relative to other members of the chain, after a long time (seven to ten half-lives of the longest-lived progeny species), all the members of the chain will be in secular equilibrium and decaying with the half-life of Species A, and all having the same activity as Species A. An important example is the ^{226}Ra decay series (Figure 3.11).

Activity Peaking Times Under General Case

Radium-226 is itself produced by other species that ultimately lead back to ^{238}U; we show this shortly. But considering only ^{226}Ra and its progeny for the moment, we note that ^{226}Ra decays with a very long half-life (about 1600 years) to ^{222}Rn, which has relatively a very short half-life, about 3.8 days. So after about 30–40 days, ^{222}Rn will be in equilibrium with ^{226}Ra. All of the progeny down to ^{210}Pb are even more short-lived, and so will rapidly come into equilibrium with ^{222}Rn, which in turn is in equilibrium with ^{226}Ra, so all of these species will have the same activity as ^{226}Ra, and will demonstrate a 1600 year half-life. If the species are also decaying for around 200 years or so, all of the progeny including and beyond ^{210}Pb will also be in equilibrium.

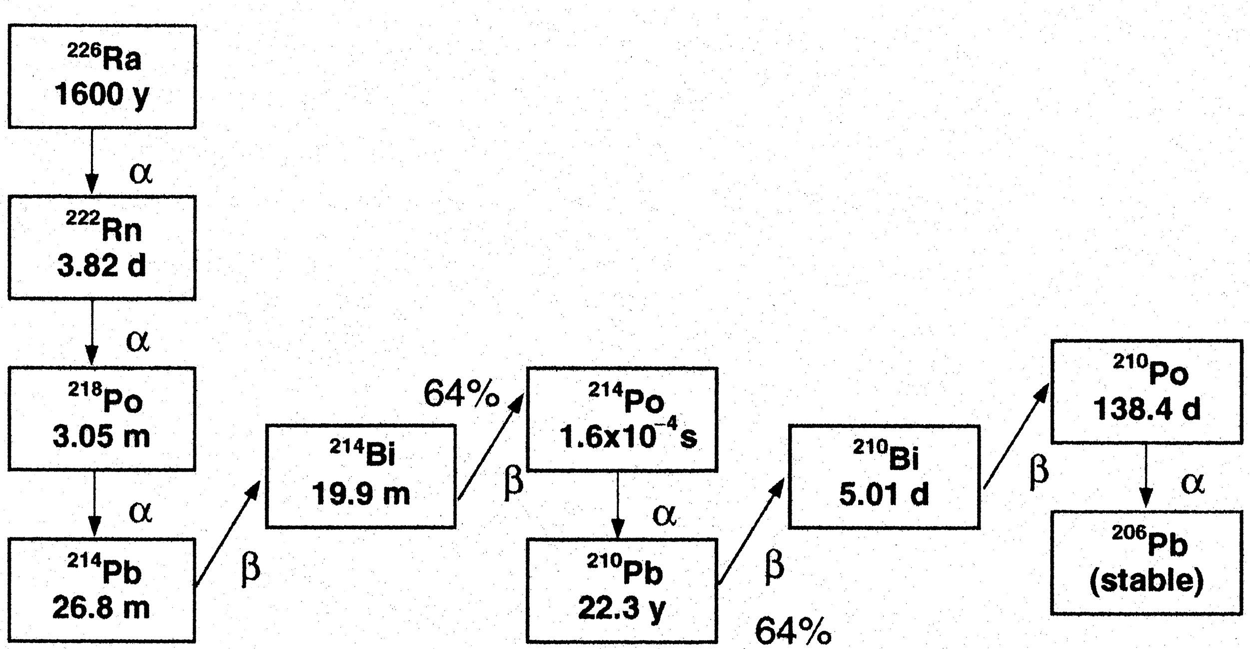
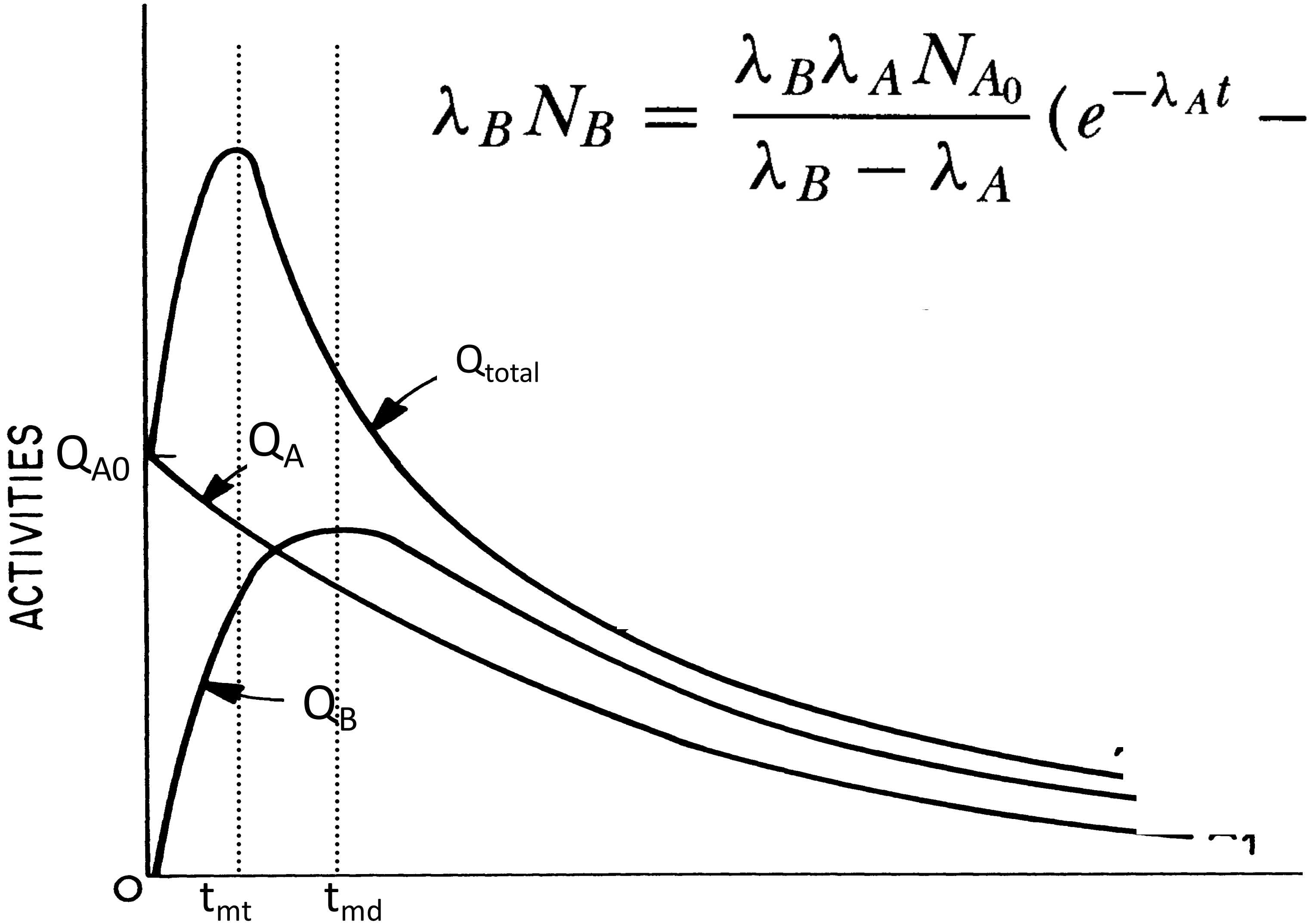


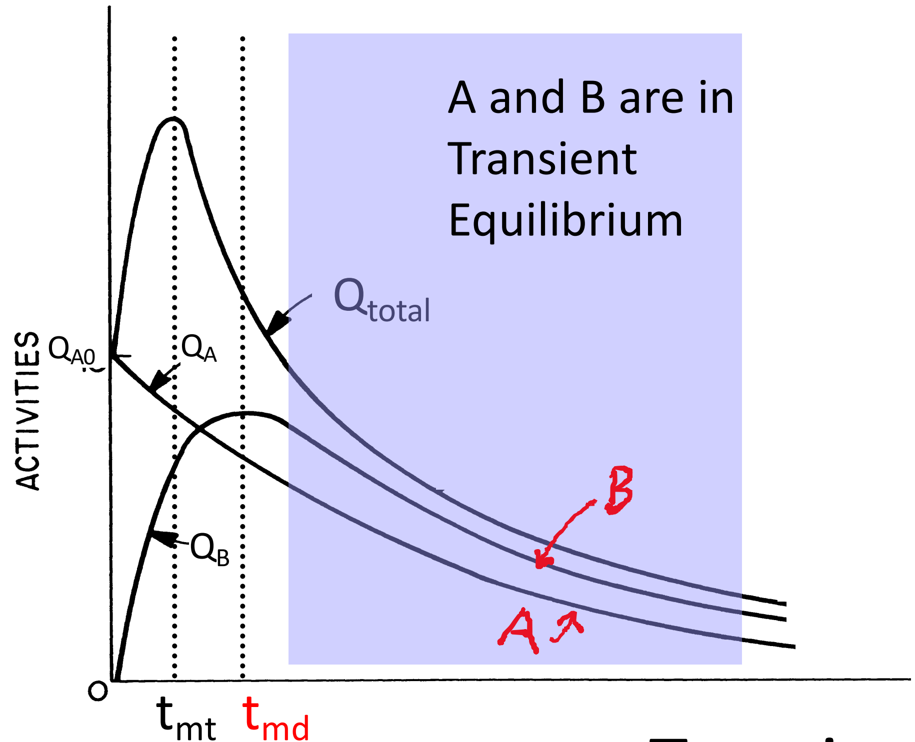
Figure 3.11 The ^{226}Ra decay series.

Activity Peaking Time Under General Case

$$\lambda_B N_B = \frac{\lambda_B \lambda_A N_{A_0}}{\lambda_B - \lambda_A}\left(e^{-\lambda_A t} - e^{-\lambda_B t}\right)$$



Transient Equilibrium: $T_A \geq T_B$ ($\lambda_A \leq \lambda_B$)



General case

$$\lambda_B N_B = \frac{\lambda_B \lambda_A N_{A_0}}{\lambda_B - \lambda_A}(e^{-\lambda_A t} - e^{-\lambda_B t})$$

Transient Equilibrium

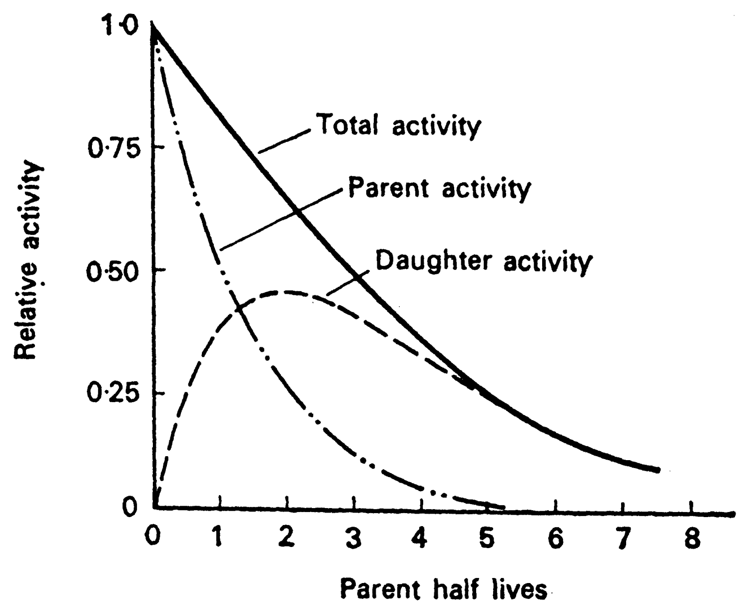
$$\lambda_B N_B = \frac{\lambda_B \lambda_A N_A}{\lambda_B - \lambda_A}$$

$$Q_B = \frac{\lambda_B}{\lambda_B - \lambda_A} Q_A$$

No Equilibrium: When $T_A < T_B$ and $\lambda_A > \lambda_B$

- The half-life of the daughter exceeds that of the parent, no equilibrium is possible.
- The number of parent atoms gradually decay to zero.
- The activity of the daughter rises to the maximum and then decays at its own characteristic rate.

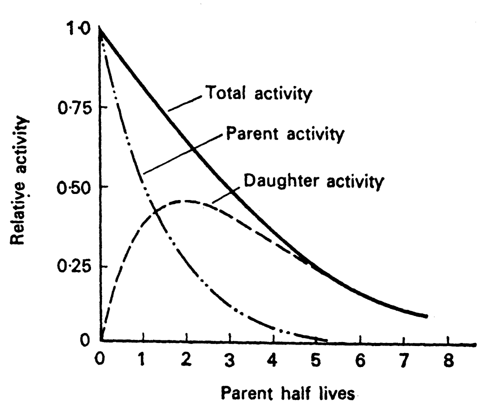
$$^{90}_{36}\mathrm{Kr} \xrightarrow[33\ \mathrm{s}]{\beta} {}^{90}_{37}\mathrm{Rb} \xrightarrow[2.74\ \mathrm{min}]{\beta} {}^{90}_{38}\mathrm{Sr} \xrightarrow[28.8\ \mathrm{years}]{\beta} {}^{90}_{39}\mathrm{Y} \xrightarrow[64.2\ \mathrm{h}]{\beta} {}^{90}_{40}\mathrm{Zr}.$$

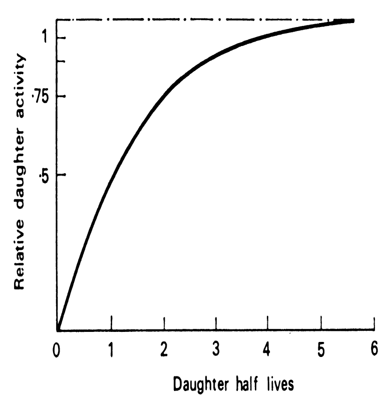


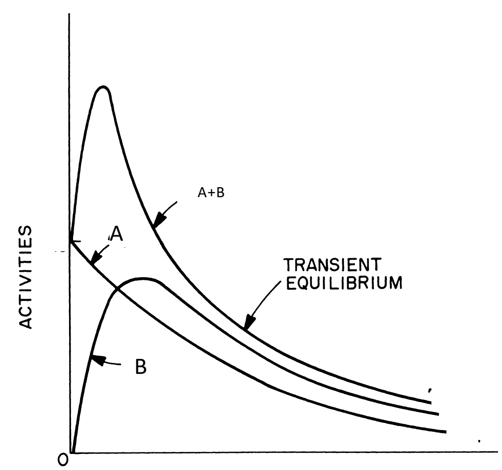
Summary of Serial Transformations

$$A \xrightarrow{\lambda_A} B \xrightarrow{\lambda_B} C,$$

General case	Secular Equilibrium	Transient Equilibrium	No Equilibrium
$T_A > T_B$	$T_A >> T_B$, $t > 7T_B$	$T_A \geq T_B$ $t > T_{md}$	$T_A < T_B$
$N_B = \frac{\lambda_A N_{A_0}}{\lambda_B - \lambda_A}(e^{-\lambda_A t} - e^{-\lambda_B t})$	$Q_B = Q_A(1 - e^{-\lambda_B t})$,	$Q_B = \frac{\lambda_B}{\lambda_B - \lambda_A} Q_A$	---







An Example

A sample contains 1 mCi of ^{191}Os at time $t = 0$. The isotope decays by β^- emission into metastable ^{191m}Ir, which then decays by γ emission into ^{191}Ir. The decay and half-lives can be represented by writing

$$^{191}_{76}\text{Os} \xrightarrow[15.4\ \text{d}]{\beta^-} {}^{191m}_{77}\text{Ir} \xrightarrow[4.94\ \text{s}]{\gamma} {}^{191}_{77}\text{Ir}.$$

(c) How many atoms of ^{191m}Ir decay between $t = 100$ s and $t = 102$ s?

(d) How many atoms of ^{191m}Ir decay between $t = 30$ d and $t = 40$ d?

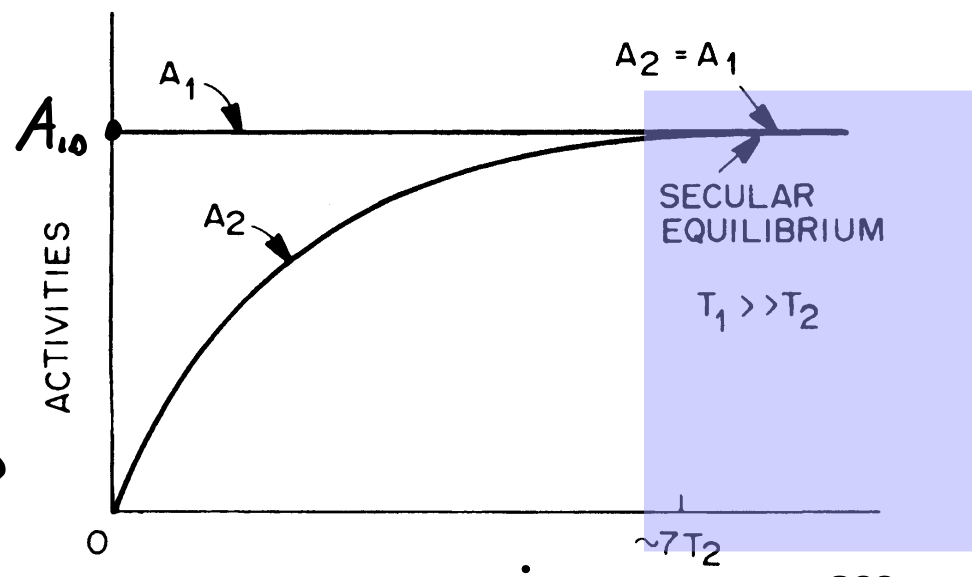
Secular Equilibrium: $T_A >> T_B$

(c) How many atoms of ^{191m}Ir decay between $t = 100$ s and $t = 102$ s?

$$^{191}_{76}\text{Os} \xrightarrow[15.4\ \text{d}]{\beta^-} {}^{191\text{m}}_{77}\text{Ir} \xrightarrow[4.94\ \text{s}]{\gamma} {}^{191}_{77}\text{Ir}.$$

Since (i) $T_A >> T_B$, and (ii) t=100-102s is longer than 7 times T_B, we are looking at a secular equilibrium …

Therefore the activity from ^{191m}Ir is roughly equal to the activity from a constant number of ^{191}Os.



$$A_2 = A_1(1 - e^{\lambda_2 t}) \Rightarrow A_2 \approx A_1 \approx A_{1,0}$$

Secular Equilibrium: $T_A >> T_B$

$$^{191}_{76}\text{Os} \xrightarrow[15.4\ \text{d}]{\beta^-} {}^{191m}_{77}\text{Ir} \xrightarrow[4.94\ \text{s}]{\gamma} {}^{191}_{77}\text{Ir}.$$

Since A_{Os} = 1 mCi, then $A_{Ir} \cong$ 1 mCi.

$$A_{Ir} = A_{Os} \approx 1mCi$$

Therefore, the number of ^{191m}Ir decayed between 100s and 102s is

$$N \approx A_{Ir} \cdot \Delta t = 1\text{mCi} \times 2second$$
$$= 2 \times 3.7 \times 10^7 = 7.4 \times 10^7.$$

A sample contains 1 mCi of ^{191}Os at time $t = 0$. The isotope decays by β^- emission into metastable ^{191m}Ir, which then decays by γ emission into ^{191}Ir. The decay and half-lives can be represented by writing

$$^{191}_{76}\text{Os} \xrightarrow[15.4\ \text{d}]{\beta^-} {}^{191\text{m}}_{77}\text{Ir} \xrightarrow[4.94\ \text{s}]{\gamma} {}^{191}_{77}\text{Ir}.$$

(d) How many atoms of ^{191m}Ir decay between $t = 30$ d and $t = 40$ d?

Solution:

Similar to question (c), there is a secular equilibrium between Os and Ir, so

$$A_{Ir}(t) \approx A_{Os}(t) \approx 1mCi.$$

Therefore, the number of Ir-191m atoms decayed is equal to the integral of Os-191 activity during the specific time interval

$$N = \int_{30d}^{40d} A_{Ir}(t) \cdot dt \approx \int_{30d}^{40d} A_{Os}(t) \cdot dt$$

$$= \int_{30d}^{40d} A_{Os}(t=0) \cdot e^{-\frac{0.693}{T}t} \cdot dt = \int_{30d}^{40d} 3.7\times10^7 \cdot e^{-\frac{0.693}{15.4d}t} \cdot dt = 7.73\times10^7.$$