Lectures 32 to 34 Objectives and Highlights

Lecture 32: In Lecture 26 we developed the full Fourier series of a real-valued 2L-periodic function f(x) and we used it to approximate and to represent f(x).

First Objective: To develop a complex version of the Fourier series of a real-valued function of period 2L and to relate this version to the real version. The complex Fourier series is often used in place of the full Fourier series. The complex Fourier series is defined as follows:

1. Definition: Given a real-valued 2L-periodic function f(x), the <u>complex Fourier series of f(x)</u> of period 2L is:

$$f(x) \approx \sum_{k=-\infty}^{+\infty} c_k \mathrm{Exp}\left[\frac{k\,\pi\,x}{L}\,i\right] \text{ where } \mathrm{Exp}[s] = e^s$$

$$\text{and } c_k = \frac{1}{2L}\int_{-L}^{L} f(t)\,\mathrm{Exp}\left[-\frac{k\,\pi\,t}{L}\,i\right]dt$$

2. Notice that the sum runs through all ineger values of the sum index k, both positive and negative. Because the function f(x) is 2L-periodic, we can replace the integration from -L to L by 0 to 2L or any other convenient interval of length 2L.

3. Question: Are the real and complex forms of a Fourier series of a 2L-peroidic real-valued function really different?
Answer: The real Full Fourier series of a 2L-periodic function was defined in Lecture 26 by

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{+\infty}\left[a_n \cos(\frac{n\,\pi x}{L}) + b_n \sin\left(\frac{n\,\pi x}{L}\right)\right] \text{ where}$$

$$a_k = \frac{1}{L}\int_0^{2L} f(t)\cos\left(\frac{k\pi t}{L}\right)dt \text{ and } b_k = \frac{1}{L}\int_0^{2L} f(t)\sin\left(\frac{k\pi t}{L}\right)dt$$

Although the appearance of the real and complex forms are quite different, the two series are actually just rearrangements of one another. More specifically, Euler's formula can be used to show that the coefficients of the two series are related as follows:

$$c_0 = \frac{a_0}{2};\quad c_k = \frac{1}{2}(a_k - i\,b_k) \text{ for } k>1;\quad c_{-k} = \frac{1}{2}(a_k + i\,b_k) \text{ for } k>1$$

$$a_0 = 2\,c_0;\quad a_k = c_k + c_{-k} \text{ and } b_k = i\,(c_k - c_{-k}) \text{ for } k>1$$

(See the calculations between Audio Clips #1 and #2 for the details.)

Second Objective: To extend the Full Fourier series representation and approximation of real-valued 2L-periodic functions to suitably restrticted non-periodic functions f(x) defined for all real numbers.

The suitable restriction for this purpose is to require that f(x) "dies off fast enough as x approaches +∞ or - ∞ so that the improper integral of |f(x)| converges to a finite number. This restriction is summarized by saying that f(x) is <u>absolutely integrable</u>.

4. Question: How do you know whether a given function is absolutely integarable?
Answer: One group of absolutely integrable functions that is easy to recognize are those that are equal to 0 for all x outside of some finite interval

The extension mentioned in the Second Objective replaces the infinite Full Forier series of a 2L-periodic function f(x) with an (convergent improper) integral from - ∞ to +∞. This extension is called the Fourier Integral Theorem, which can be stated precisely as follows:

The Fourier Integral Theorem: Suppose that f(x) is an absolutely integrable and piecewise continuous function defined for all real numbers x from from - ∞ to +∞. Then at points of continuity of f(x), the integral (*):

$$(*) \quad \int_0^{+\infty} \left[A(w)\cos(w\,x) + B(w)\sin(w\,x) \right] dw \quad \text{where}$$

$$A(w) = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(t)\cos(w\,t)\,dt \quad \text{and}$$

$$B(w) = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(t)\sin(w\,t)\,dt$$

converges to the value f(x). At any jump discontiuity x = c of f(x), the integral (*) converges to $\frac{f(c+)+f(c-)}{2}$, the average of the right and left hand limits of f at x = c.

5. Summary of the derivation of the derivation of the Fourier Integral Theorem:

Recall from Lecture 26 or from Page 1 of this summary, that the Full Fourier series of a 2L-periodic function f(x) is:

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{+\infty}\left[a_n\cos\!\left(\frac{n\,\pi x}{L}\right) + b_1\sin\!\left(\frac{n\,\pi x}{L}\right)\right] \quad where$$

$$a_k = \frac{1}{L}\int_0^{2L} f(t)\cos\!\left(\frac{k\pi t}{L}\right)dt \quad and \quad b_k = \frac{1}{L}\int_0^{2L} f(t)\sin\!\left(\frac{k\pi t}{L}\right)dt$$

We can replace the interval of integration [0, 2L] by any interval of length 2L in computing the Fourier series coefficients because f(x) is 2L-periodic. Let’s change the interval of integration to [-L, L] and insert the integrals for the coefficients into the series to obtain:

$$\frac{1}{2L}\int_{-L}^{L} f(t)\,dt + \frac{1}{L}\sum_{n=1}^{+\infty}\left[\int_{-L}^{L} f(t)\cos\!\left(\frac{n\pi t}{L}\right)dt\,\cos\!\left(\frac{n\,\pi x}{L}\right) + \int_{-L}^{L} f(t)\sin\!\left(\frac{n\pi t}{L}\right)dt\,\sin\!\left(\frac{n\,\pi x}{L}\right)\right]$$

To obtain a representation of an absolutely integrable non-periodic function f(x) for all real numbers x, we let $w_n = \frac{n\,\pi}{L}$ in the preceding expression to obtain:

$$\frac{1}{2L}\int_{-L}^{L} f(t)\, dt + \frac{1}{L}\sum_{n=1}^{+\infty}\left[\int_{-L}^{L} f(t)\cos\left(w_n t\right)dt\, \cos(w_n x) + \int_{-L}^{L} f(t)\sin\left(w_n t\right)dt\,\sin\left(w_n x\right)\right]$$

The points $w_n = \frac{n\,\pi}{L}$ are equally spaced along the number line at a distance $\Delta w = w_{n+1} - w_n = \frac{\pi}{L}$ between successive points. If we let L increase without bound, then the first integral approaches 0 (because f(t) is absolutely integrable) and Δw approaches 0 and the preceding sum is a convergent approximation for the integral:

$$\frac{1}{\pi}\int_{0}^{+\infty}\left[\cos(w\ x)\int_{-\infty}^{+\infty} f(t)\cos(w\ t)\, dt + \sin(w\ x)\int_{-\infty}^{+\infty} f(t)\sin(w\ t)\, dt\right]dw\ ,$$

which is equivalent to (*) in the statement of the Fourier Integral Theorem.

The Fourier Integral Theorem can be used to approximate a function by replacing the infinite limits of integration in (*) by suitable finite values as in the Example just before Audio Clip #5.

The example after Audio Clip #5 shows how the Fourier Integral representation can be used to model and solve a boundary value problem involving heat flow on a long, thin metal bar.

6. Question: What is the practical difference between using Fourier series or Fourier integrals for the solution of boundary value problems?

Answer: The boundary value problem is solved by the method of separation of variables applied to the homogeneous conditions only both in Lecture 32 and in Lectures 27 through 31. However, there is one major difference in the way things turn out in the application in this lecture: In the separation of variables applications in Lectures 28 through 31, at least one of the separated single-vararible systems turned out to be a Sturm-liouville system. This enabled us to find a complete sequence of eigenvalues $\{\lambda_n\}$ and eigenfunctions $\{G_n\}$ for the Sturm-Liouville system. Then we used the corresponding eigenfunctions to find a finite sum or infinite series of the eigenfunctions $\{u_n = G_n\ H_n\}$ of the S-L system to satisfy the remaining non-homogeneous condition(s).

In the example discussed between Audio Clip #5 and Audio Clip #8, and others like it, we get instead product form solutions $u_\alpha(x,t) = F_\alpha(x)\ G_\alpha(t)$ for each positive real number α (not just for a sequence of distinct values). Finally, we use an integral from 0 to $+\infty$ to combine these product form solutions of the homogeneous conditions into a solution that satisfies the non-homogeneous conditions too.

7. Complex Version of the Fourier Integral Theorem: Suppose that f(x) is an absolutely integrable and piecewise continuous function defined for all real numbers x from - ∞ to +∞. Then at points of continuity of f(x), the integral (**):

$$(**)\quad \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty} f(t)\,\mathrm{Exp}(-w\ t\ i)\,dt\right]\mathrm{Exp}(w\ t\ i)\,dw$$

converges to the value f(x). At any jump discontiuity x = c of f(x), the integral (*) converges to $\frac{f(c+)+f(c-)}{2}$, the average of the right and left hand limits of f at x = c.

This version of the Fourier Integral Theorem follows directly from the complex version of the Full Fourier series of a 2L-periodic function f(x) discussed in Lecture 32 as L approaches +∞ in exactly the same way as the real version of the Fourier Integral Theorem follows from the real version of this series. (see the Summary of this derivation on Page 2.)

8. My expectations concerning the Fourier Integral and its applications:
As a result of taking this course, you should:
i) Know what kind of functions have Fourier integral representations,
ii) Be able to compute this representation for simple absolutely integrable functions,
iii) Be able to set up a boundary-value problem such as the example after Audio Clip #5 and recognize that the Fourier integral is appropriate for its solution.
However, we did not spend enough time on the the Fourier Integral Theorem to expect you to find the Fourier Integral representation of more complicated functions or to be able to make simplifications of Fourier Integral solutions of boundary value problems such as those in the latter part of the the example after Audio Clip #5.

9. Question: What is the Complex Fourier Transform and how is it computed and applied?
Answer: If f(x) is an absolutely integrable, piecewise continuous real-valued function defined for all real numbers, then the complex version of the Fourier Integral Theorem guarantees that the integral

$$g[w] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(t)\,\mathrm{Exp}[-w\ t\ i]\,dt$$

defines a function g[w] with the property that

$$f[x] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} g(w)\,\mathrm{Exp}[w\ x\ i]\,dw$$

at all contiuity points of f. The function g[w] is called the **complex Fourier transform** of f(x) and is denoted by g[w] = $\Im$[f(x)], while the integral

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} g(w)\,\mathrm{Exp}[w\ x\ i]\,dw$$

is called the **inverse complex Fourier Transform** of g[w] and is denoted by $\Im^{-1}$[g[w]]. The two absolutely integrable functions f(x) and g(w) are caled a c**omplex Fourier Transform pair.**

A short table of Fourier Transform pairs is provided provided betwee Lecture 32 to Lecture 33 under the Lectures button on your course CD or on our course web site. Not all engineering books and software programs define the complex Fourier transform in exactly the same way but the choice we have made is probably the most common. It also has the advantage that the the Fourier transform pairs can be used in both directions; that is, the complex Fourier transform of the function on the right in the pair is the function on the left in the pair.

Examples 1 and 2 after Audio Clip #3 in Lecture 33 illustrate the calculation of Fourier transforms for relatively simple functions. Example 2 is particularly interesting because it shows that for positive constant K and α, the functions

$$f(x) = K\ \mathrm{Exp}[-\alpha\ x^2] \ \text{ and } \ g(w) = K\ \sqrt{\frac{1}{2\alpha}}\mathrm{Exp}\left[-\frac{w^2}{4\alpha}\right]$$

are a fourier transform pair. Both f and g have the same functional form and are called **Gaussian**s because, in the special case in which their integrals are equal to 1, they are **Gaussian probability distributions.** The graph of a Gaussian is like the bell-shaped curves that represent normal probability distributions. When α is large, the graph is narrow and peaked; when α is small, the bell-shaped graph is relatively flat and spread out. The calculation above shows that the Fourier transform of a sharply peaked Gaussian is another Gaussian that is relatively flattened and vice-versa. This fact is related to the Heisenberg Uncertainty Principle as explained just before Just Do It! 33.2.

The main operational properties of the complex Fourier Transform are derived between Audio Clips #5 and #6 in Lecture 33. These and other operational properties are listed in Formulas 28 through 42 in the Table of Fourier Transforms that we have provide on the course CD and web site. The application discussed between Audio Clips #6 and #7 in Lecture 33 shows how the linearity and derivative properties can be used to find a particular solution of a second order non-homogeneous linear differential equation.

At the beginning of Lecture 34, the complex Fourier transfoms of some important special functions are computed including the so-called Dirac delta function and truncated periodic functions. The point of these examples is to show how unknown Fourier transforms of functions can be computed through approximation by functions with known Fourier transforms.

10. Question: What are the Fourier Sine and Cosine Transforms, how are they related to the complex Fourier Transform, and how are they computed and applied?

Answer:

11. Definitions: If f(x) is a piecewise contiuous, absolutely integrable function defined on the interval $0 < x < +\infty$, then:

$$g_c(w) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} f(t) \cos(w\ t)\, dt$$

is the Fourier Cosine transform of f(x), while

$$g_s(w) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} f(t) \sin(w\ t)\, dt$$

is the Fourier Sine transform of f(x).

The Fourier Cosine Transform $g_C(w)$ is also denoted by $\Im_C[f(x)]$, while $\Im_S[f(x)]$ also stands for the Fourier Sine transform $g_S(w)$.

12. Given a piecewise contiuous, absolutely integrable function f(x) defined on the interval $0 < x < +\infty$, the Fourier Cosine transform of f(x) is just the Complex Fourier Transform of the even extension of f(x) to $-\infty < x < 0$, while the Fourier Sine Transform of f(x) is just the Complex Fourier Transform of the odd extension of f(x) to $-\infty < x < 0$. (See the discussion after Audio Clip #5 in Lecture 34 for details.)

The Fourier Sine and Cosine transforms are well suited to representing and approximating functions that are defined on the half-infinite interval $0 < x < +\infty$. However, their use in this context can be avoided by applying the Complex Fourier Transform to the even or odd extension of f(x) to $-\infty < x < 0$.

13. The Fourier Sine and Cosine transforms have operational properties similar to those of the Complex Fourier Transform. Some of the most important of these properties are developed betwee Audio Clips #5 and #6 in Lecture 34 and a more complete list is given in Formulas 20 through 42 of our Fourier Transform Tables. The remainder of Lecture 34 is devoted to the discussion of several example of calclations and applications of the Fourier Sine and Cosine Transforms.

Lectures 35 to 38 Objectives and Highlights

Lecture 35: Earlier in Lecture 26 we developed the full Fourier series of a 2L-periodic function f(x) and we used it to approximate and to represent f(x). The hardest part of computing Fourier series is doing the integration that is required for the coefficients of the Fourier series of f(x).

First Objective: To describe an approximation for a 2L-periodic function f(x) that is similar to Full Fourier series of f(x) truncated after m terms and that does not require the calculation of integrals but rather only "sampling " values of f(x) at a finite number of points along an interval of length 2L. We call that approximation the **discrete real Fourier polynomial for f(x) of order m**.

Background: In Lecture 26, the initial terms of the Full Fourier series of f(x) was defined by

$$\frac{a_0}{2}+\left[a_1\cos(\frac{\pi x}{L})+b_1\sin\left(\frac{\pi x}{L}\right)\right]+.....+\left[a_m\cos(\frac{m\pi x}{L})+b_m\sin\left(\frac{m\pi x}{L}\right)\right]$$

$$where\;\; a_k=\frac{1}{L}\int_0^{2L} f(t)\cos\left(\frac{k\pi t}{L}\right)dt \;\; and \;\; b_k=\frac{1}{L}\int_0^{2L} f(t)\sin\left(\frac{k\pi t}{L}\right)dt$$

Note that this is a *finite sum* with 2m+1 terms, not an infinite series, and each of its terms has a factor of either $\cos(\frac{k\pi x}{L})$ *or* $\sin(\frac{k\pi x}{L})$. For these reasons, this function is often called the *continuous real Fourier polynomial of order m for f(x)*. Its graph approximates the graph of f(x) and the approximation gets better and better as the order m increases. Its computation requires the calculation of definite integrals.

In Lecture 35 we develop a corresponding *discrete real Fourier polynomial of order m*:

$$p_{m,L}(x)=\frac{A_0}{2}+\left[A_1\cos(\frac{\pi x}{L})+B_1\sin\left(\frac{\pi x}{L}\right)\right]+.....+\left[A_m\cos(\frac{m\pi x}{L})+B_m\sin\left(\frac{m\pi x}{L}\right)\right]$$

whose graph approximates that of f(x) and whose coefficients $A_0, A_1, B_1, ..., A_m, B_m$ can be computed using only the values of f(x) at the following 2m+1 equally spaced points on the interval [0, 2L]:

$$x_0=0,\, x_1=\frac{2L}{2m+1},\, x_2=\frac{4L}{2m+1},\, ...,x_{2m-1}=\frac{(2m-1)L}{2m+1},\, x_{2m}=\frac{2mL}{2m+1}$$

and the following formulas:

$$A_0=\frac{2}{2m+1}[\,f(x_0)+f(x_1)+.....+f(x_{2m})] \quad and \;\; for\;\; k=1,....,m:$$

$$A_k=\frac{2}{2m+1}[\,f(x_0)\cos(\frac{\pi x_k}{L})+f(x_1)\cos(\frac{2\pi x_k}{L})+.....+f(x_{2m})\cos(\frac{(2m)\pi x_k}{L})$$

$$B_k=\frac{2}{2m+1}[\,f(x_0)\sin(\frac{\pi x_k}{L})+f(x_1)\sin(\frac{2\pi x_k}{L})+.....+f(x_{2m})\sin(\frac{(2m)\pi x_k}{L})$$

Other Lecture 35 Highlights:
1. Where do these coefficient formulas come from?
Answer: They are the consequence of the orthogonality of the sine and cosine functions for the "inner product" $\left(f(x), g(x)\right) = f(x_0)g(x_0) + f(x_1)g(x_1) + \dots\dots\dots f(x_{2m})g(x_{2m})$. (See discussion between Audio Clips #2 and #3 in Lecture 35 for details.

2. Question: For a given function f(x) and a given order m, are the continuous and discrete real Fourier polynomials the same?
Answer: No, their graphs are similar but not identical. However, for most practical purposes, the discrete real Fourier polynomial of order m can be used instead of the continuous real Fourier polynomial of order m. (See discussion between Audio Clips #6 and #7 in Lecture 35 for details.

3. The discrete real Fourier polynomial of order m can be viewed as the least squares solution of the system of linear equations obtained by equating the values of a trigonometric polynomial of order m to the values of f(x) at the 2m+1 equally space points for the inner product in item 1 above. (See discussion between Audio Clips #5 and #6 in Lecture 35 for details.

Lecture 36: In Lecture 32, we developed the complex form of the full Fourier series of a 2L-periodic function f(x), and then we showed that it is just a rearrangement of the real Full Fourier Series of f(x).

Second Objective: To develop a discrete complex Fourier polynomial of order n that provides an approximation for a 2L-periodic function f(x) so that when n = 2m+1, the discrete complex Fourier polynomial is just a rearrangement of the discrete real Fourier polynomial of order m.

We develop the *discrete complex Fourier polynomial of order 2m+1 for a 2L-periodic function f(x)* by using the same set of 2m+1 equally spaced points in the interval [0, 2L] and then defining

$$q_{2m+1}(x) = C_0 + C_1 Exp[\frac{\pi x}{L} i] + C_2 Exp[\frac{2\pi x}{L} i] + \dots\dots + C_{2m} Exp[\frac{2m\pi x}{L} i] \;\; where \;\; for \;\; p = 0, \dots, 2m,$$

$$C_p = \frac{1}{2m+1}\left[f(x_0) + f(x_1)Exp[-\frac{\pi p}{2m+1}] + f(x_2)Exp[-\frac{2\pi p}{2m+1}] + \dots\dots + f(x_{2m})Exp[-\frac{2m\pi p}{2m+1}] \right]$$

1. The coefficients of the discrete real and complex Fourier polynomials of order 2m+1 are just rearrangements of one another as is shown between Audio Clips #4 and #5 in Lecture 36.
2. More generally, for any positive integer n, the *discrete complex Fourier polynomial of order n for a 2L-periodic function f(x)* can be defined by using n equally spaced points

$$x_0 = 0, \;\; x_1 = \frac{2L}{m}, \;\; x_2 = \frac{4L}{m}, \dots\dots\dots, x_{m-1} = \frac{2(m-1)L}{m}$$

in the interval [0, 2L] as follows:

$$q_{m,L}(x) = C_0 + C_1 Exp[\frac{\pi x}{L} i] + C_2 Exp[\frac{2\pi x}{L} i] + ... + C_{m-1} Exp[\frac{(m-1)\pi x}{L} i] \;\; where \;\; for \;\; p = 0,....,m,$$

$$C_p = \frac{1}{m}\left[\, f(x_0) + f(x_1)Exp[-\frac{\pi p}{m} i] + f(x_2)Exp[-\frac{2\pi p}{m} i] + ... + f(x_{m-1})Exp[-\frac{(m-1)\pi p}{m} i]\right]$$

3. **Computing discrete complex Fourier polynomial coefficients with the Fourier Transfom Matrix.** For any integer n > 1 and any real or complex number w, there are exactly n complex numbers z such that the nth-power of z is w. For w = 1, these n complex numbers are called *the nth-roots of unity* and they are spaced evenly around the circle of radius 1 centered at the origin. The successive nth roots of unity are $1, \omega_n, \omega_n^{\,2}, \omega_n^{\,3}, ..., \omega_n^{\,n-1}$.
(See the discussion between Audio Clips #5 and #6 in Lecture 36.)

4. The coefficients $\vec{C} = \begin{bmatrix} C_0 \\ C_1 \\ . \\ . \\ C_{m-2} \\ .C_{m-1} \end{bmatrix}$ and the function values $\vec{f} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ \\ \\ f(x_{m-2}) \\ f(x_{m-1}) \end{bmatrix}$ are related by the matrix equation $\vec{f} = fourier[n]\vec{C}$ where fourier[n] is the n by n matrix whose: first row is all 1s, whose second row is the successive complex nth-roots of unity: $1, \omega_n, \omega_n^{\,2}, \omega_n^{\,3}, ..., \omega_n^{\,n-1}$, whose third row is the squares of the nth-roots of unity $1, \omega_n^{\,2}, \omega_n^{\,4}, \omega_n^{\,6}, ..., \omega_n^{\,2(n-1)}$, and so on. The columns of the matrix fourier[n] are mutually orthogonal for the inner product $(u, v) = u_1\overline{v_1} + u_2\overline{v_2} + ... + u_n\overline{v_n}$ and its inverse is given by $fourier[n]^{-1} = \frac{1}{n}\overline{fourier[n]}$. (See the discussion after Audio Clip #6 in Lecture 36.)

Lecture 37: At the end of Lecture 36 we established the following relationship between the vector of values of a 2L-periodic function f(x) and the vector of discrete complex Fourier polynomial coefficients of f(x) :

$$\vec{f} = fourier[n]\vec{C} \;\; and \;\; \vec{C} = \frac{1}{n}\overline{fourier[n]}\,\vec{f}$$

1**. Discrete Fourier Transforms (DFT)**: More generally, if **x** and **y** are any two vectors with n components and if **y** = fourier[n] **x**, then ***y*** *is the Discrete Fourier Transform (DFT) of* ***x*** and **x** is the *Inverse Discrete Fourier Transform(IDFT) of* ***y***.
(See the discussion and examples between Audio Clips #1 and #2 in Lecture 37).

2. **Discrete Fourier Convolution (DFC)** If **x** = $\{x_0, x_1, ...x_{n-1}\}$ and **y** = $\{y_0, y_1, ...y_{n-1}\}$ are any two vectors with n components, then the *Discrete Fourier Convolution of x and y is given by* ***z=x*y***= $\{z_0, z_1, ...z_{n-1}\}$ where $z_r = \sum x_p y_q$ *where* $p+q=r$ *or* $p+q=r+n$

3. Discrete Fourier Convolution **z=x*y** can also be computed by matrix multiplication **x*y= C_x y** where $\mathbf{C_x}$ is the n by n matrix whose first row vector is x, whose second row of $\mathbf{C_x}$ is the vector obtained by "rotating" **x** to the left one spot, whose third row of $\mathbf{C_x}$ is vector obtained by "rotating" x to the left two spots, and so on. (See the discussion after Audio Clip #5 in Lecture 37.)

4. One important application of DFC to computer arithmetic is discussed between Audio Clips #6 and #7 in Lecture 37. There are other applications. It turns out that DFC **z=x*y** corresponds to term-by-term multiplication of the discrete Fourier transforms of **x** and **y**. More precisely, , the *Convolution Rule* holds:

If **u** = fourier[n]$^{-1}$ **x** and **v** = fourier[n]$^{-1}$ **y**, then **x*y=**fourier[n] (**u**×**v**)

(See the discussion and examples after Audio Clip #6 in Lecture 37.)

5. The number of arithmetic operations required to compute the DFC of two vectors **x** and **y** with n components or the DFT of a vector with n components is of the order of n^2. For large values of n (as often happens in applications), this makes these computations slow and computationally expensive. (See the discussion at the end of Lecture 37.)

Lecture 38:

Third Objective: To explain *the mathematical basis of the Fast Fourier Transform (FFT).* The FFT is an algorithm for calculating the Discrete Fourier Transform of a vector **x** with n components or the Discrete Fourier Convolution of two vectors **x** and **y** with n components with an operation count of the order of n Log(n).

1. **The mathematical basis of the FFT:** Given a positive integer m, let n = 2m. Then the squares of the successive nth-roots of unity are the successive mth-roots of unity covered twice. (This simple fact is what does it all!)

2. **The FFT Reduction Procedure from n = 2m to m:** Given a vector with n=2m components u = $\{u_0, u_1, u_2, u_3, ...u_{2n-2}, u_{2n-1}\}$, split u into two vectors

$$u^{(1)} = \{u_0, u_2, ...u_{2n-2}\} \ \ and \ \ u^{(2)} = \{u_1, u_3, ...u_{2n-1}\}$$

and let

$$v^{(1)} = \ fourier[m]\{u_0, u_2, ...u_{2n-2}\} \ \ and \ \ v^{(2)} = \ fourie[m]\{u_1, u_3, ...u_{2n-1}\}$$

Then the vector **v** = fourier[n] **u** can be recovered from $\mathbf{v^{(1)}}$ and $\mathbf{v^{(2)}}$ with the *reconstruction equations*:

1) $v_p = v_p^{(1)} + \omega_n^{\ p} v_p^{(2)} \quad for \ p = 0,1,...,m-1$

2) $v_{p+m} = v_p^{(1)} - \omega_n^{\ p} v_p^{(2)} \quad for \ p = 0,1,...,m-1$

(See the discussion and examples after Audio Clip #2 for an explanation of why this works.)

3. **The FFT Algorithm**: To compute the DFT vector **v** = fourier[n**]** **u** where n = 2^s for some positive integer s, proceed as follows:

Splitting Phase: Split **u** into two vectors $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$ with 2^{s-1} components each as in the FFT Reduction Procedure. Then split the two vectors $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$ into four vectors $\mathbf{u}^{(1,1)}$ and $\mathbf{u}^{(1, 2)}$for $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2,1)}$ and $\mathbf{u}^{(2,2)}$ for $\mathbf{u}^{(2)}$ each with 2^{s-2} components. Continue this splitting process until you have vectors of length 2.

Reconstruction Phase: Apply the 2 by 2 matrix fourier[2] to each of the vectors at the end of the splitting phase and then use the reconstruction equations to construct the fourier[4] transforms of each of the vectors in the next to the last step of the Splitting Phase. Repeat this procedure for each of these vectors of length 4 to obtain the fourier[8] transforms of 2^{s-4}vectors, and so on until **v** = fourier[n**]** u is obtained.
(See the discussion and examples between Audio Clips #3 and #4.)

4. The operation count for computing the DFT of a vector of length n or the DFC of two vectors of length n is of the order of n^2 from the definitions but of the order of n Log(n) for the FFT. This difference in efficiency is huge for large n. (See the discussion after Audio Clip #5 in Lecture 38.)