

ECE 586GT: Problem Set 2: Problems and Solutions

Uniqueness of Nash equilibria, zero sum games, evolutionary dynamics

Due: Wednesday, September 23, 11:59 p.m.**Reading:** Course notes, Sections 1.5-1.7 and Chapter 2

1. [Some continuous games]

Consider a normal form game with a finite number of players n and strategy sets $S_i = [0, 1]$ for $1 \leq i \leq n$.

- (a) Suppose $u_i(x) = x_i + \frac{b}{2}x_{tot}^2$ for some constant b , where $x_{tot} = x_1 + \dots + x_n$. Identify all the pure Nash equilibria (i.e. all Nash equilibria in pure strategies). Hint: Consider cases $b \geq 0$, $-\frac{1}{n} \leq b < 0$ and $b \leq -\frac{1}{n}$ separately.

Solution: If $b \geq 0$ then $u_i(x_i, x_{-i})$ is strictly increasing in x_i so that $x_i = 1$ is a strictly dominant strategy for all i . So the all one vector $\mathbf{1}$ is the unique Nash equilibrium.

Suppose for the rest of this solution that $b < 0$. We begin by deriving the best response $B_i(x_{-i})$ for each i . Since u_i is strictly concave as a function of x_i there is a unique best response of player i for each x_{-i} so we understand $B_i(x_{-i})$ as the best response itself, rather than as a set of best responses. We find $\frac{\partial u_i}{\partial x_i} = 1 + bx_{tot} = 1 + bx_i + b \sum_{j:j \neq i} x_j$. So the graph of u_i as a function of x_i is a downward facing parabola with peak at $-\frac{1}{b} - \sum_{j:j \neq i} x_j$. Therefore, taking into account the constraint $x_i \in [0, 1]$,

$$B_i(x_{-i}) = \begin{cases} 0 & \text{if } -\frac{1}{b} - \sum_{j:j \neq i} x_j < 0 \\ -\frac{1}{b} - \sum_{j:j \neq i} x_j & \text{if } -\frac{1}{b} - \sum_{j:j \neq i} x_j \in [0, 1] \\ 1 & \text{if } -\frac{1}{b} - \sum_{j:j \neq i} x_j > 1 \end{cases}$$

Suppose now that x is a Nash equilibrium strategy profile. Note that $B_i(x_{-i}) > 0$ only if $x_{tot} \leq -\frac{1}{b}$. So if $x_{tot} > -\frac{1}{b}$ then $x_i = 0$ for all i implying $x_{tot} = 0$. This contradiction shows it is impossible to have $x_{tot} > -\frac{1}{b}$. So $x_{tot} \leq -\frac{1}{b}$ for any Nash equilibrium. Therefore the first case in the expression for $B_i(x_{-i})$ doesn't arise for any player i .

In the second case for $B_i(x_{-i})$ the best response value for x_i makes $x_{tot} = -\frac{1}{b}$, and then we can conclude that the second case arises for all of the players. This is possible if and only if $-\frac{1}{b} \leq n$ or, equivalently, $b \leq -\frac{1}{n}$. Therefore in case $b \leq -\frac{1}{n}$ the set of pure Nash equilibria is $\{x \in S : x_{tot} = -\frac{1}{b}\}$.

In the third case for $B_i(x_{-i})$ the best response value for x_i makes $x_{tot} < -\frac{1}{b}$, and then we can conclude that the third case arises for all of the players, which is possible if and only if $-\frac{1}{b} \geq n$ or, equivalently, $-\frac{1}{n} \leq b < 0$. Therefore, in case $-\frac{1}{n} \leq b < 0$, the all one vector $\mathbf{1}$ is the unique pure Nash equilibrium.

In summary, by combining cases of $b \geq 0$ and $b < 0$, we get that if $b \geq -\frac{1}{n}$ then $x^* = \mathbf{1}$ is that unique Nash equilibrium and if $b < -\frac{1}{n}$ then the set of pure Nash equilibria is $\{x \in S : x_{tot} = -\frac{1}{b}\}$.

- (b) More generally, suppose $u_i(x) = x_i + \frac{1}{2}x^T A x$ for a given symmetric matrix A while still assuming $S_i = [0, 1]$ for all i . Under what conditions on A does the Debreu-Glicksberg-Fan theorem imply that there exists a pure Nash equilibrium? (Find the most general condition possible.)

Solution: The strategy sets are compact and the payoff functions are continuous. Concavity of $u_i(x_i, x_{-i})$ with respect to the scalar value x_i is equivalent to $A_{ii} \leq 0$ for all i . So the DGF theorem applies if and only if the diagonal entries of A are less than or equal to zero.

- (c) For the payoff functions of part (b), under what conditions on A does Rosen's theorem imply that there is at most one pure Nash equilibrium?

Solution: Since $\nabla_{x_i} u_i(x) = 1 + \sum_j A_{ij} x_j$ we find that for strategy profiles $\bar{x}$ and x^* , the sum in the DSC condition is given by

$$\begin{aligned} & \sum_i (\bar{x}_i - x_i^*) \left(1 + \sum_j A_{ij} x_j^* \right) + \sum_i (x_i^* - \bar{x}_i) \left(1 + \sum_j A_{ij} \bar{x}_j \right) \\ &= (\bar{x} - x^*)^T A x^* + (x^* - \bar{x}) A \bar{x} \\ &= -(\bar{x} - x^*)^T A (\bar{x} - x^*). \end{aligned}$$

Therefore the DSC condition becomes $(\bar{x} - x^*)^T A (\bar{x} - x^*) < 0$ for all pairs of nonidentical strategy profiles $\bar{x}$ and x^* . Thus, the DSC condition is equivalent to A being negative definite.

The example in part (a) corresponds to $A = \mathbf{1}\mathbf{1}^T$, the all ones matrix, which is negative semidefinite. The example shows that uniqueness does not follow if A is only negative semidefinite.

2. [Min-max and max-min strategies for rock-scissors-paper game]

Consider the rock-scissors-paper game described in Example 1.17. Let $\ell(p, q)$, with $p, q \in \{R, S, P\}$, denote the payoff function of player 2, which player 1 seeks to minimize.

- (a) Assuming both players use pure strategies, identify the set of min-max strategies of player 1 and the set of max-min strategies of player 2. Determine whether there is a duality gap. If no gap, determine the set of all saddle points in pure strategies.

Solution: For example, taking $p = R$ we find

$$\max_q \ell(R, q) = \ell(R, P) = 1.$$

More generally, $\max_q \ell(p, q) = 1$ for any $p \in \{R, S, P\}$.

Thus $\min_p \max_q \ell(p, q) = 1$, and R, S, P are all min-max strategies. So $\{R, S, P\}$ is the set of min-max strategies for player 1.

Similarly, $\{R, S, P\}$ is the set of max-min strategies for player 2, yielding a max-min value of -1 . Therefore there is a duality gap of size $1 - (-1) = 2$ and no saddle point.

- (b) Repeat part (a) but now consider p, q in the space of mixed strategies.

Solution: Note that

$$\ell(p, q) = q_R(p_S - p_P) + q_P(p_R - p_S) + q_S(p_P - p_R).$$

For fixed p , the expression is maximized with respect to q by putting all mass on the largest coefficient of q_R, q_P , or q_S . Therefore

$$\max_q \ell(p, q) = \max \left\{ p_S - p_P, p_R - p_S, p_P - p_R \right\}.$$

Hence the min-max strategies of player 1 are

$$\bar{p} = \arg \min_p \max \left\{ p_S - p_P, p_R - p_S, p_P - p_R \right\}.$$

Since

$$(p_S - p_P) + (p_R - p_S) + (p_P - p_R) = 0,$$

the maximum of the three quantities must be greater than or equal to zero. Equality is achieved if and only if $\bar{p}_R = \bar{p}_S = \bar{p}_P$, or $\bar{p} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. The min-max value is therefore 0.

Similarly, the max-min value is 0, achieved by $\bar{q} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Therefore there is no duality gap. Since $\bar{p}$ is the unique min-max strategy and $\bar{q}$ is the unique max-min strategy, $(\bar{p}, \bar{q})$ is the unique saddle point.

3. [Karush-Kuhn-Tucker Debreu-Glicksberg-Fan]

Consider the convex optimization problem: $\min x$ over $x \in \mathbb{R}$ s.t. $g(x) \leq 0$ where $g(x) = x^2 - \epsilon$.

- (a) Suppose $\epsilon > 0$. Is the relaxed Slater condition satisfied? What is the Lagrangian $L(x, \lambda)$? Find (x^*, λ) satisfying the KKT conditions, if possible. Does strong duality hold in the sense that $\inf_x \sup_{\lambda \geq 0} L(x, \lambda) = \sup_{\lambda \geq 0} \inf_x L(x, \lambda)$? If $L(x, \lambda)$ is viewed as the objective function for a two-person zero-sum game between primal and dual players, does the DGF theorem for existence of Nash equilibrium apply?

Solution: The relaxed Slater condition holds because, for example, $x = 0$ is a feasible point and $g(0) < 0$. The Lagrangian is given by $L(x, \lambda) = x + \lambda(x^2 - \epsilon)$. Since $\frac{\partial}{\partial x} L(x, \lambda) = 1 + 2\lambda x$, the KKT conditions are $1 + 2\lambda x^* = 0$, $(x^*)^2 - \epsilon \leq 0$, and $\lambda((x^*)^2 - \epsilon) = 0$. The unique solution is $(x^*, \lambda^*) = (-\sqrt{\epsilon}, \frac{1}{2\sqrt{\epsilon}})$. Yes, strong duality holds. It holds in theory by the KKT necessary conditions for optimality and it can also be checked explicitly as follows. On the one hand, $\inf_x L(x, \lambda)$ is achieved at $x = -\frac{1}{2\lambda}$ yielding $\inf_x L(x, \lambda) = -\frac{1}{4\lambda} - \lambda\epsilon$ and $-\frac{1}{4\lambda} - \lambda\epsilon$ is maximized over $\lambda \geq 0$ at $\lambda^* = \frac{1}{2\sqrt{\epsilon}}$, achieving the max min value $-\sqrt{\epsilon}$. On the other hand,

$$\sup_{\lambda \geq 0} L(x, \lambda) = \begin{cases} x & \text{if } |x| \leq \sqrt{\epsilon} \\ +\infty & \text{else} \end{cases}$$

so that $\sup_{\lambda \geq 0} L(x, \lambda)$ is minimized over x at $x^* = -\sqrt{\epsilon}$, achieving the min sup value $-\sqrt{\epsilon}$. So the max min and min sup values are both $-\sqrt{\epsilon}$.

The DGF theorem does not apply because the strategy sets are not compact: x ranges over the noncompact set $\mathbb{R}$ and λ ranges over the noncompact set $[0, +\infty)$. (Nevertheless the Nash equilibrium exists and is given by (x^*, λ)).

- (b) Repeat part (a) for $\epsilon = 0$.

Solution: The relaxed Slater condition fails because there is no feasible x such that $g(x) < 0$. As in part (a) but with $\epsilon = 0$, the Lagrangian is given by $L(x, \lambda) = x + \lambda x^2$ and the KKT conditions are $1 + 2\lambda x^* = 0$, $(x^*)^2 \leq 0$, and $\lambda(x^*)^2 = 0$. There is no (x^*, λ) satisfying the KKT conditions.

Checking for strong duality: On the one hand, for $\lambda > 0$, $\inf_x L(x, \lambda)$ is achieved at $x = -\frac{1}{2\lambda}$ yielding $\inf_x L(x, \lambda) = -\frac{1}{4\lambda}$ for $\lambda > 0$ (for $\lambda = 0$, $\inf_x L(x, \lambda) = -\infty$.) and the

supremum of $-\frac{1}{4\lambda}$ is asymptotically achieved over $\lambda > 0$ as $\lambda \rightarrow \infty$, yielding the sup min value 0. On the other hand,

$$\sup_{\lambda \geq 0} L(x, \lambda) = \begin{cases} 0 & \text{if } x = 0 \\ +\infty & \text{else} \end{cases}$$

so that $\sup_{\lambda \geq 0} L(x, \lambda)$ is minimized over x at $x = 0$, achieving the min sup value 0. So yes, strong duality still holds in the sense of the problem statement. The DGF theorem does not apply because the strategy sets are not compact as in part (a).

- (c) If $\epsilon = 0$ and we were to impose the constraints $x, \lambda \in [-1, 1]$ then would there exist a Nash equilibrium for the two-person zero-sum game based on the Lagrangian? Briefly explain your answer.

Solution: Yes, because then the DGF conditions would apply. Or we can easily check that $(x^*, \lambda) = (-\frac{1}{2}, 1)$ is a Nash equilibrium (it is also unique).

4. [Dual of a transport problem]

Let $n \geq 1$ and let W be an $n \times n$ matrix with nonnegative elements. Consider the following linear optimization problem:

$$\begin{aligned} \min_X \quad & \sum_{i,j \in [n]} X_{i,j} W_{i,j} \\ \text{subject to:} \quad & \sum_{j \in [n]} X_{i,j} = 1 && \text{for } i \in [n] \\ & \sum_{i \in [n]} X_{i,j} = 1 && \text{for } j \in [n] \\ & X_{i,j} \geq 0 && \text{for } i, j \in [n] \end{aligned}$$

An interpretation is that there are n sources of some good, n destinations, $W_{i,j}$ is the cost of transporting a unit of good from i to j , and $X_{i,j}$ is the amount of good transported from i to j . Each source provides one unit of good and each destination receives one unit of good. The problem in vector matrix form can be written as:

$$\begin{aligned} \min_X \quad & \langle X, W \rangle \\ \text{subject to:} \quad & X\mathbf{1} = \mathbf{1}, \quad X^T\mathbf{1} = \mathbf{1}, \quad X \geq 0. \end{aligned}$$

- (a) Derive the dual problem by first finding the Lagrangian function. Find a simple version of the dual problem. (Hint: You can either eliminate the Lagrange multipliers for the constraints $X_{i,j} \geq 0$, or don't handle those constraints with Lagrange multipliers in the first place.

Solution: Using strong duality for linear programs, we find

$$\begin{aligned}
& \min_{X: X\mathbf{1}=\mathbf{1}, X^T\mathbf{1}=\mathbf{1}, X \geq 0} \langle X, W \rangle \\
&= \min_{X: X \geq 0} \max_{\mu^{(1)}, \mu^{(2)}} \langle X, W \rangle + (\mathbf{1} - X\mathbf{1})^T \mu^{(1)} + (\mathbf{1} - X^T\mathbf{1})^T \mu^{(2)} \\
&= \max_{\mu^{(1)}, \mu^{(2)}} \min_{X: X \geq 0} \langle X, W \rangle + (\mathbf{1} - X\mathbf{1})^T \mu^{(1)} + (\mathbf{1} - X^T\mathbf{1})^T \mu^{(2)} \\
&= \max_{\mu^{(1)}, \mu^{(2)}} \min_{X: X \geq 0} \mathbf{1}^T (\mu^{(1)} + \mu^{(2)}) + \langle X, W - \mu^{(1)}\mathbf{1}^T - \mathbf{1}(\mu^{(2)})^T \rangle \\
&= \max_{\mu^{(1)}, \mu^{(2)}: \mu^{(1)}\mathbf{1}^T + \mathbf{1}(\mu^{(2)})^T \leq W} \mathbf{1}^T (\mu^{(1)} + \mu^{(2)})
\end{aligned}$$

In other words the dual problem is

$$\begin{aligned}
& \max_{\mu^{(1)}, \mu^{(2)}} \sum_{i \in [n]} \mu_i^{(1)} + \sum_{j \in [n]} \mu_j^{(2)} \\
& \text{subject to:} \quad \mu_i^{(1)} + \mu_j^{(2)} \leq W_{i,j} \quad \text{for } i, j \in [n]
\end{aligned}$$

(An interpretation of the dual variables is that $\mu_i^{(1)}$ is a transport charge per unit good at source i and $\mu_j^{(2)}$ is a transport charge per unit good at destination j . The dual problem entails maximizing the sum of payments subject to a constraint showing the charges are justified by the transport costs. The equality constraints in the primal problem could be replaced by inequality constraints: $X\mathbf{1} \geq \mathbf{1}$, $X^T\mathbf{1} \geq \mathbf{1}$, and that would lead to nonnegativity constraints on the multipliers in the dual problem.

- (b) Find the common value of the primal and dual, and solutions, for $W = \begin{pmatrix} 3 & 8 \\ 2 & 4 \end{pmatrix}$.

Solution: $X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\mu^{(1)} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, $\mu^{(2)} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$, with value 7 each. (The solution to the dual problem is not unique. Adding any constant multiple of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ to $\mu^{(1)}$ and subtracting it from $\mu^{(2)}$ yields another solution.)

5. [Evolutionarily stable strategies and states]

Consider the following symmetric, two-player game::

	1	2
1	4,4	1,6
2	6,1	0,0

- (a) Does either player have a (weakly or strongly) dominant strategy?

Solution: No.

- (b) Identify all the pure strategy and mixed strategy Nash equilibria.

Solution: (1,2) and (2,1) are pure strategy NE. As usual, there is no NE in which only one strategy is pure. If (p, q) is an NE such that both p and q are nondegenerate mixed strategies, either action of player one must be a best response to q , so $4q_1 + 1 - q_1 = 6q_1$ or $q = (\frac{1}{3}, \frac{2}{3})$. Similarly, $p = (\frac{1}{3}, \frac{2}{3})$. Thus, $((\frac{1}{3}, \frac{2}{3}), (\frac{1}{3}, \frac{2}{3}))$ is the unique NE in nondegenerate mixed strategies.

- (c) Identify all evolutionarily stable pure strategies and all evolutionarily stable mixed strategies.

Solution: Recall that if p is an ESS, then (p, p) is an NE. The pure strategy NEs found in part (b) are not symmetric, so there are no pure ESSs. It remains to see whether the mixed strategy NE p given by $p = (\frac{1}{3}, \frac{2}{3})$ is an ESS. By definition, we need to check whether for any $p' \neq p$, either (i) $u(p', p) < u(p, p)$, or (ii) ($u(p', p) = u(p, p)$ and $u(p, p') > u(p', p')$). Since any pure strategy is a best response to p , any mixed strategy is also a best response to p , so $u(p', p) = u(p, p)$ for all choices of p' . The question thus comes down to whether $u(p, p') > u(p', p')$ for all $p' \neq p$. That is, whether $\frac{1}{3}(4p'_1 + 1 - p'_1) + \frac{2}{3}(6p'_1) > p'_1(4p'_1 + 1 - p'_1) + (1 - p'_1)(6p'_1)$ or equivalently whether $5p'_1 + \frac{1}{3} > 7p'_1 - 3p'^2_1$ which is equivalent to $3(p'_1 - \frac{1}{3})^2 > 0$. This condition is true for all $p' \neq p$, so p is an ESS.

- (d) The replicator dynamics based on this game represents a large population consisting of type 1 and type 2 individuals. Show that the evolution of the population share vector $\theta(t)$ under the replicator dynamics for this model reduces to a one dimensional ordinary differential equation for $\theta_t(1)$, the fraction of the population that is type 1.

Solution: The replicator dynamics for the population share vector θ_t are given by $\dot{\theta}_t(a) = \theta_t(a)(u(a, \theta_t) - u(\theta_t, \theta_t))$ for $a \in \{1, 2\}$. Although there are two equations, this system is actually one dimensional because $\theta_t(2) = 1 - \theta_t(1)$. Let $x_t = \theta_t(1)$. Then $u(1, x_t) = 4x_t + 1 - x_t = 3x_t + 1$ and $u(x_t, x_t) = 4x_t^2 + 7x_t(1 - x_t) = 7x_t - 3x_t^2$. Thus $u(1, x_t) - u(x_t, x_t) = 3x_t^2 - 4x_t + 1 = 3(x_t - 1)(x_t - 1/3)$. So the replicator dynamics become

$$\dot{x}_t = 3x_t(x_t - 1)(x_t - 1/3). \quad (1)$$

- (e) Identify the steady states of the replicator dynamics.

Solution: The right hand side of (1) is zero for $x_t \in \{0, \frac{1}{3}, 1\}$, so there are three steady states for the replicator dynamics: $(1, 0)$, $(\frac{1}{3}, \frac{2}{3})$, and $(0, 1)$.

- (f) Of the steady states identified in the previous part, which are asymptotically stable states of the replicator dynamics? Justify your answer.

Solution: Of the three steady states, only $(\frac{1}{3}, \frac{2}{3})$ is asymptotically stable. If $x_0 = \epsilon$ for an arbitrarily small but positive ϵ , then $\dot{x}_t > 0$ and x will converge monotonically up to $\frac{1}{3}$, so 0 is not even a stable steady state (so it is not asymptotically stable). Similarly, 1 is not a stable steady state. However, $\frac{1}{3}$ is an asymptotically stable steady state of x because the right hand side of (1) has a down crossing of zero at $\frac{1}{3}$. Hence $(\frac{1}{3}, \frac{2}{3})$ is an asymptotically stable state for the replicator dynamics. (Another justification for this problem can be given by applying general facts about ESSs and replicator dynamics. States $(1, 0)$ and $(0, 1)$ can't be asymptotically stable, or even stable, states for the replicator dynamics, because, when played against themselves, they don't give NEs. The mixed state $(\frac{1}{3}, \frac{2}{3})$ is an asymptotically stable state of the replicator dynamics because it is an ESS.)