ECE 586GT: Problem Set 6
Optimal selling mechanisms, and cooperative games

Due: Wednesday, December 2, 11:59 p.m.
Reading: Course notes, Sections 7.2 and Chapter 8.

1. [**Meyerson winner selection rule**]
Suppose Myerson's optimal selling mechanism is used by a seller to sell an object to a set of three bidders, using the min-to-win payment rule. Suppose the virtual valuation functions are given by $\psi_1(x_1) = x_1 - 1$, $\psi_2(x_2) = x_2 - 2$, and $\psi_3(x_3) = x_3 - 3$, and the value of the object to seller is zero (i.e. free disposal assumption holds). For each of the bid profiles given below, determine which bidder, if any, gets the object, and for what payment.

   (a) Bid profile $(6.1, 5.8, 7.9)$.

   (b) Bid profile $(0.5, 1.9, 1.6)$.

2. [**Revenue optimal sequential selling mechanism**]
Suppose there is a population of n bidders and one seller with an object to sell. The value of the object to bidder i is X_i, where $X_1, \ldots, X_n$ are independent, and each is exponentially distributed with parameter λ. Each bidder i knows his own value, and knows the probability distribution of the other values. The seller knows the probability distribution of all values.

   This problem addresses the Bayes-Nash implementation of a revenue optimal seller mechanism, subject to the following protocol. The bidders appear before the seller one at a time in the fixed order $1, 2, \ldots, n$. When a bidder appears, he submits a bid to the seller, and, on the basis of the bid, the seller must either accept or reject the bid. If the bid is accepted, the bidder wins the object and makes a payment. If the bid is rejected, the bidder has no further chance to win and the seller moves on to the next bidder. The mechanism is also constrained to be individually rational. The seller has the option of not selling the object to any bidder, in which case the value of the object to the seller is zero.

   (a) Briefly describe what the revelation principle implies in the context of this game, and why it applies.

   (b) Calculate the virtual valuation function $\psi(x_i) = x_i - \frac{1-F_i(x_i)}{f_i(x_i)}$ of a buyer i. (By symmetry, answer does not depend on i.)

   (c) Identify the optimal mechanism in this context for $n = 1$, and calculate the resulting expected revenue R_1^*.

   (d) Let R_n^* denote the maximum expected revenue for the game with n bidders. Describe the actions of the seller for the first bidder in terms of R_{n-1}^*, and find a recursion expressing R_n^* in terms of R_{n-1}^*.

3. **[Welfare optimal vs. revenue optimal auction]**
Consider the sale of a single object, assuming the value of the object to the seller is $r = 0$ (free disposal) and there are 2 bidders with independent valuations X_1, X_2. Suppose X_1, the valuation of bidder one, is uniformly distributed over the interval $[a, b]$, where $0 \leq a < b$ and X_2 is uniformly distributed over the interval $[0, b]$.

   (a) Identify an IC, IR selling mechanism that maximizes the social welfare, and determine the expected payment to the seller as a function of a, b.

   (b) Identify an IC, IR selling mechanism that maximizes the expected payment to the seller, based on the known distributions of the values described above. Assume the two submitted bids are in the interval $[0, b]$. (You don't need to calculate the expected revenue.)

   (c) Identify the limit of the expected revenue for the mechanisms of parts (a) and (b) as $a \to b$ for fixed $b > 0$. Give a simple explanation for your answers.

4. **[Proof of Bondareva-Shapley theorem]**
Complete the proof of the Bondareva-Shapley theorem in Section 8.1 of the notes by showing that the dual of the problem (P) given in the proof is the problem (D) stated in the theorem. (Please refer to November 7 version of notes or later.)

5. **[Identification of cohesion and core]**
The cooperative games below are for a set of four players, $I = \{1, 2, 3, 4\}$. As usual, assume transferrable utilities. Simplify your answers as much as possible.

   (a) Suppose $v_1 = (v_1(S))_{S \subset I}$ is given by

$$v_1(S) = \begin{cases} 1 & \text{if } \{1,2\} \subset S \text{ or } \{3,4\} \subset S \\ 0 & \text{else} \end{cases} .$$

   Is (I, v_1) cohesive? Find the core of (I, v_1).

   (b) Suppose $v = v_1 + v_2$, where v_1 is defined in (a), and

$$v_2(S) = \begin{cases} 1 & \text{if } \{1,3\} \subset S \text{ or } \{2,4\} \subset S \\ 0 & \text{else} \end{cases} .$$

   For example, the players could be working for two days, with values v_1 the first day and v_2 the second day. Is (I, v) cohesive? Find the core of (I, v).

6. **[A market with transferable utilities]**
Consider the market with transferable utilities $M = (I, \ell, (w_i), (f_i))$ such that $I = \{0, 1, \ldots, m\}$ for some $m \geq 1$, $\ell = 1$, and

$$\begin{aligned} f_0(z_0) &= z_0^{1/2} & w_0 &= 0 \\ f_i(z_i) &= 0 & w_i &= 1, \text{ for } i \neq 0 \end{aligned}$$

   Player 0 represents a factory owner and the other players represent workers, each with one unit of work to contribute.

   (a) Write down the core constraints for the induced game and give a simple expression for the core.

   (b) Find the competitive equilibrium, (z^*, p^*), for the induced cooperative game, and identify the corresponding payoff profile x^*.