**University of Illinois** **Fall 2026**

# ECE 586GT: Problem Set 5

## Games with incomplete information, multistage games, VCG mechanisims

**Due:** Wednesday, November 11, 11:59 p.m.
**Reading:** Course notes, Section 5.3, Chapters 6, & Section 7.1

1. **[Half Kuhn poker]**
Consider the following two-player zero-sum game. There is a deck of three cards, $\{L, M, H\}$ with the ordering $L < M < H$. The players are randomly assigned one card each with each possibility having probability 1/6. The players see their own cards but not the card of the other player. If the players simply compared cards, the player with the larger valued card would win 1 unit of money. Instead, the players follow the following protocol. Player 1 first says "raise" or "check." If player 1 says check the two players show their cards and the player with the larger valued care wins one unit. If player 1 says check then player 2 can either meet or fold. If player 2 meets then the stakes are doubled and the player with the higher card wins 2 units. If player 2 folds then player 1 wins one unit.

   (a) Which player has an advantage over the other and why?

   (b) Make a sketch of the game tree. (Hint: Try making it as useful as you can by giving meaningful names to nodes or information sets. To avoid clutter, since it is a zero sum game you could label the leaves with the payoff of player 1 which is equal to the loss of player 2, instead of giving the payoffs of both players. It might be simpler to first have nature select a card for player 1 and then, if needed, later select a card for player 2, equally likely to be either card not selected for player 1.

   (c) Identify a subgame perfect equilibrium for the game and the corresponding value of the game. Is the subgame perfect equilibrium unique?

2. **[The lemon problem]**
A seller has a used car to sell and a buyer is interested in buying it. Suppose the value $\theta$ of the car to seller is random and uniformly distributed on the interval $[0, 1]$. The seller knows $\theta$, which can be thought of as the type of the seller. The buyer does not know $\theta$ but it is assumed that the value of the car to the buyer is $a + b\theta$, for some constants $a$ and $b$ such that $a \geq 0$ and $1 < a + b < 2$. These constraints ensure that the car is worth more to the buyer than to the seller for any $\theta$, so, payment aside, the socially optimal outcome would be for the car to be transferred from seller to buyer. But can they agree on a price? Suppose the seller, knowing $\theta$, calculates a reserve price $r = r(\theta)$, which is the minimum amount the seller would sell the car for. Suppose the buyer selects a price $p \in [0, 1]$ and offers to pay $p$ for the car. If $p \geq r$, the car is sold for price $p$. Otherwise the car is not sold. The payoff for the seller is $u_s(p, r, \theta) = \theta \mathbf{1}_{\{p<r\}} + p\mathbf{1}_{\{p\geq r\}}$ and the payoff for the buyer is $u_b(p, r, \theta) = (a + b\theta - p)\mathbf{1}_{\{p\geq r\}}$.

   (a) Identify a weakly dominant strategy for the seller.

   (b) Assuming seller uses its weakly dominant strategy, find the price $p$ the buyer should offer to maximize her/his expected payoff. (Suppose the buyer knows a, b, and the distribution of $\theta$).

   (c) What happens in the special case $a = 0$ and $1 < b < 2$. Give a simple explanation.

3. **[Repeated play for the Bertrand equilibrium problem]**
Suppose $n$ players, for some $n \geq 2$, represent firms that can each produce a common divisible good at a cost $c$ per unit of good. Suppose the action of each player is to declare a price $p_i$ per unit of good. Suppose there is an aggregate demand of consumers such that if the lowest price offered by any player is $p_{min}$ then the consumers purchase a total quantity $(a - p_{\min})_+$ of good, where $a$ is a constant with $a > c$, and they purchase an equal amount from each player offering the minimum price. The game is among the players offering prices; the consumers are not considered to be part of the game. As we saw in problem set 1, problem 7(a), $p$ is a Nash equilibrium if and only if $p_{\min} = c$ and $p_i = c$ for at least two players. The resulting payoff vector is the all zero vector. Basically, the competition among the players drives the sum of payoffs to zero at Nash equilibrium.

   (a) Let's see if positive payoffs can be sustained in the infinite repeated game with the Bertrand based game as a stage game, with weight $(1-\delta)\delta^{t-1}$ on stage $t$. To begin, fix $\bar{p}$ with $c < \bar{p} < \frac{a+c}{2}$, and consider the trigger strategy based on the script that every player bids price $\bar{p}$ in every stage. Players follow the script, with switching to playing $c$ forever if any player deviated from the script in an earlier stage. Each player will receive payoff $(a - \bar{p})(\bar{p} - c)/n$ per stage. Find the minimum $\delta$ so that such strategy profile is subgame perfect.

   (b) Identify the set of payoff vectors $v$ in the Nash realization region for the infinite repeated game with stage game based on the Bertrand selling mechanism.

4. **[VCG applied to combinatorial auction]**
Suppose a VCG mechanism is applied to sell the objects in $\mathcal{O} = \{a, b, c\}$ to three bidders, who have submitted the following bids:

$$\begin{array}{lll} v_1\{a\} = 12, & v_1\{a,b\} = 14 & \\ v_2\{b,c\} = 6, & v_2\{a,b,c\} = 12 & \\ v_3\{b\} = 2, & v_3\{c\} = 5, & v_3\{b,c\} = 6. \end{array}$$

Assume the value of any bundle to any bidder, if not listed above, is zero. Also, assume that for any bidder, at most one of the bidder's bids may be used. In particular, by bidding such that $v_3\{b,c\} < v_3\{b\} + v_3\{c\}$, bidder 3 communicates that $b$ and $c$ are partial substitutes for the bidder. Suppose also that the seller has a reserve price, or value, of 1, for each item that is not sold. Determine the assignment of objects to bidders and the payments of the bidders, for the VCG mechanism under truthful bidding. Use the individually rational version of payments.

5. **[Supporting a public good with private information]**
Consider $n$ players, any of whom could perform a particular job that would benefit all the players. For example, the players could be participating in some venture and the job is to make preparations to ensure safety in case of some unanticipated event. Suppose each player derives value 1 if at least one player does the job. Suppose $\theta_i$ with $0 \leq \theta_i \leq 1$ is private information of player $i$, which represents the cost for player $i$ to do the job. So player $i$ knows the value of $\theta_i$ but neither the coordinator nor any other player knows the value of $\theta_i$.

   (a) Explain how this situation could be handled by a VCG mechanism. In particular, describe the bids made by the player to the coordinator, how the coordinator selects which player to do the job, and how much the player should be paid by the coordinator.

(b) Your solution to part does not need to be budget balanced in the sense that the coordinator may need to subsidize the game by paying a player to do the job. A simple attempt at fixing the problem would be for the coordinator impose a fee on each player in the amount of $\sigma/n$. In such way the coordinator has a balanced budget. Are the other important properties of the VCG mechanism preserved: maximum social welfare, optimality of truthful bidding, nonnegative total payoffs for all players?