Name: _____ **NetID:** _____

ECE 585GT Fall 2026 Exam 1, October 22, 2026

90 min; 1 page notes 2-sided OK; closed book; no calculators

1. **[30 points]** Consider the two-player game with payoff matrix shown, where player 1 selects a letter for a row and player 2 selects a number for a column:

	1	2	3
A	6,4	7,7	4,3
B	8,5	5,2	6,4
C	7,6	3,4	5,5

   (a) (5 points) Identify all dominant strategies and all weakly dominant strategies by either player.
   Solution: Neither player has a dominant or weakly dominant strategy.

   (b) (5 points) Identify all pairs of strategies such that both strategies are for the same player and one strategy dominates or weakly dominates the other.
   Solution: B dominates C, 1 dominates 3

   (c) (5 points) Identify the pure Nash equilibria.
   Solution: There are two pure NE: (A,2) and (B,1)

   (d) (5 points) Identify the Nash equilibria involving nondegenerate mixed strategies, if any.
   Solution: Any NE must assign zero probability to dominated strategies, so we seek an NE of the form $((a, 1-a, 0), (b, 1-b, 0))$ with $0 < a < 1$ and $0 < b < 1$. Both A and B should be a best response for player 1, requiring $6b + 7(1-b) = 8b + 5(1-b)$. or $7 - b = 3b + 5$ or $b = 1/2$. Both 1 and 2 should be a best response for player 2, requiring $4a + 5(1-a) = 7a + 2(1-a)$, or $a = 1/2$. (There is no NE in which only one of the strategies is a nondegenerate mixed strategy, because the responses to strategies 1 and 2 are unique.) Therefore, $((1/2, 1/2, 0), (1/2, 1/2, 0))$ is the unique NE including a nondegenerate mixed strategy.

   (e) (5 points) Is the game a potential game? If so, identify the potential function. If not, show why.
   Solution: No. For example, there is a violation in the upper 2×2 matrix because $(7-4)+(5-7) \neq (8-6)+(2-5)$.

   (f) (5 points) Consider the iterated best response algorithm. Suppose player 1 selects one of the three pure strategies, all three having equal probability. Then player 2 selects a best response to player 1. Then player 1 selects a best response to player 2. and so on. With what probability does the algorithm converge? Given the algorithm converges, what is the probability distribution of the limiting strategy pair?
   Solution: If player 1 initially selects A, then player 2 selects strategy 2, and the pair of strategies is $(A, 2)$ after all subsequent iterations. If player 1 initially selects strategy B or C, then player 2 selects strategy 1, then player 1 selects B, and the pair of strategies is $(B, 1)$ after all subsequent iterations. Thus, the sequence produced converges with probability one. The limit is $(A, 2)$ with probability $\frac{1}{3}$ and $(B, 1)$ with probability $\frac{2}{3}$.

2. **[20 points]** Determine whether each statement is TRUE or FALSE, *and give a justification for your answer* for more than half credit.

   (a) (5 points) Consider a finite two-person zero-sum game with payoff matrix A. Suppose V is the value of the game: $V = \min_p \max_q p^T A q = \max_q \min_p p^T A q$. Suppose (p^*, q^*) is a strategy profile such that $V = (p^*)^T A q^*$. Then (p^*, q^*) is a saddle point (aka Nash equilibrium).

Solution: False. For example, the matching pennies game

	1	2
1	1	−1
2	−1	1

has the unique saddle point $((0.5, 0.5), (0.5, 0.5))$ and the value of the game is $V = 0$. But $\ell((x, 1-x), (0.5, 0.5)) = V$ for any $x \in [0, 1]$.

Intuitive explanation: Picture a graph of the function $p^T Aq$. The graph has a saddle shape and the saddle point occurs in the middle of the saddle. Picture a horizontal plane that goes through the saddle point. It intersects the saddle in two curves giving many strategy profiles with value V that are not saddle points.

(b) (5 points) There exists at least one minimizer x^* of the function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f(x_1, x_2) = x_1^4 - 12x_1x_2^2 + e^{x_1^2+3x_2^2}$.

Solution: True. Because of the exponential term, $f(x) \to +\infty$ as $\|x\| \to \infty$, so we can select a constant R sufficiently large that $f(x) > f(0) = 1$ if $\|x\| > R$. Thus, whether f has a minimum value over $x \in \mathbb{R}^2$ reduces to whether f has a minimum value over the compact domain $\{x : \|x\| \leq R\}$. Such minimum exists by the Weierstrass extreme value theorem.

(c) (5 points) For a finite normal form game, the set of correlated equilibria is a convex set.

Solution: Yes, because the set of correlated equilibria is determined by a finite set of linear constraints. Equivalently, it is the intersection of a finite number of halfspaces, which must be a convex set.

(d) (5 points) For a player i in a finite normal form game, the set of maxmin strategies is a convex set.

Solution: True. Let V_i be the maxmin value for player i: $V_i = \max_{p_i} \min_{s_{-i}} u(p_i, s_{-i})$. Then a distribution p over the set of actions available to player i is maxmin optimal if and only if $u(p, s_{-i}) \geq V_i$ for all s_{-i}. Thus if p and $\overline{p}$ are both maxmin optimal, then for any choice of s_{-i} and $0 < \lambda < 1$,

$$u_i(\lambda p + (1-\lambda)\overline{p}, s_{-i}) = \lambda u_i(p, s_{-i}) + (1-\lambda)u_i(\overline{p}, s_{-i}) \geq \lambda V_i + (1-\lambda)V_i = V_i, .$$

implying that $\lambda p + (1-\lambda)\overline{p}$ is also maxmin optimal

3. **[20 points]** Recall the replicator dynamics equation for the fractional sizes of subpopulations for a given finite set of types S and fitness matrix $(u(x, y))_{x,y \in S}$, where $u(x, y)$ is the fitness of a type x individual encountering a type y individual:

$$\dot{\theta}_t(a) = \theta_t(a)(u(a, \theta_t) - u(\theta_t, \theta_t)).$$

Let $\overline{\theta}$ be a probability distribution vector indexed by S, representing a distribution of relative population sizes.

(a) (10 points) Prove or disprove the following statement: If $(\overline{\theta}, \overline{\theta})$ is a Nash equilibrium for the symmetric two player game based on u, then $\overline{\theta}$ is a steady state (aka equilibrium) for the population dynamics.

Solution: True. By definition, if $(\overline{\theta}, \overline{\theta})$ is a Nash equilibrium then $\overline{\theta}$ must be a best response to the other player playing $\overline{\theta}$. Therefore, for all a such that $\overline{\theta}(a) > 0$, a must

be a best response to $\overline{\theta}$. In particular, the value $u(a, \overline{\theta})$ is the same for all a such that $\overline{\theta}(a) > 0$, and that value must thus also equal $u(\overline{\theta}, \overline{\theta})$. Thus, $\overline{\theta}(a)(u(a, \overline{\theta}) - u(\overline{\theta}, \overline{\theta})) = 0$ for all a, so that $\overline{\theta}$ is a steady state.

(b) (10 points) Prove or disprove the following statement: If $\overline{\theta}$ is a steady state for the population dynamics then $(\overline{\theta}, \overline{\theta})$ is a Nash equilibrium for the two player symmetric game based on u.

Solution: False. For example, any choice of $\overline{\theta}$ concentrated on a single population type is a steady state, but not all pure actions give Nash equilibrium points.

4. **[30 points]** Consider the following two-player game with the two-dimensional payoff function $\ell(i, j)$ for $i, j \in \{0, 1\}$ shown:

	1	2
1	$\binom{0}{0}$	$\binom{1}{0}$
2	$\binom{0}{1}$	$\binom{0}{0}$

(a) (15 points) Consider a halfspace of the form $\mathcal{H} = \{v \in \mathbb{R}^2 : u \cdot v \leq c\}$ such that $u = \binom{u_1}{u_2}$ is a nonzero vector with $u_1 \geq 0$ and $u_2 \geq 0$. What is the minimum value of c such that $\mathcal{H}$ is approachable (in the sense of Blackwell) by player 1? Express c in terms of u as explicitly as possible.

Solution: By the lemma for approachability of a halfspace, the minimum c is the minmax value for player 1 for the scalar payoff $u \cdot \ell$. Note that $u \cdot \ell$ is given by

	1	2
1	0	u_1
2	u_2	0

Using probability distribution $(p, 1-p)$ for player 1 we have

$$c = \min_p \max_j u \cdot \ell(p, j) = \min_p \max\{u_2(1-p), u_1 p\} = \frac{u_1 u_2}{u_1 + u_2}$$

where we used the fact that the minimum is achieved by selecting p to make $u_2(1-p) = u_1 p$, or $(p, 1-p)$ proportional to (u_2, u_1).

(b) (15 points) What is the minimum value of $\delta \geq 0$ such that the square region $S_\delta = [0, \delta]^2$ is approachable by player 1. Give an explicit numerical value and show your reasoning.

Solution: For a given u as in part a, the smallest halfspace with boundary normal to u that contains S_δ is $\{v : u \cdot v \leq \delta(u_1 + u_2)\}$. Thus, by Blackwell's approachability theorem, S_δ is approachable if and only if $\frac{u_1 u_2}{u_1 + u_2} \leq \delta(u_1 + u_2)$ for all vectors u as in part a. So the smallest δ is given by $\delta = \max_u \frac{u_1 u_2}{(u_1 + u_2)^2} = \frac{1}{4}$, where the maximum us achieved when $u_1 = u_2$.