

ECE 586GT: Problem Set 6: Problems and Solutions

Optimal selling mechanisms, and cooperative games

Due: Thursday, December 6, at beginning of class

Reading: Course notes, Sections 6.2 and Chapter 7.

1. [A comparison of seller mechanisms with free disposal]

Consider a seller with a single item to sell to two bidders. The private valuation X_i of the item to bidder i is known by the seller to be exponentially distributed with parameter λ_i for $i \in \{1, 2\}$. The bidders know their own valuations. The seller has zero value for the item (free disposal) and is not required to sell the item. For each of the following seller mechanisms, describe the allocation and payment rules, and find the expected payoff of the seller. The mechanisms should be individually rational and incentive compatible.

- (a) Myerson's mechanism that maximizes the seller's expected payoff. Assume the min-to-win form of payment rule. (Hint: The area rule for expectations could be useful: $\mathbb{E}[Y] = \int_0^\infty \mathbb{P}\{Y > t\} dt - \int_{-\infty}^0 \mathbb{P}\{Y \leq t\} dt$.)

Solution: For a given x_i , the virtual valuation is $\psi_i(x_i) = x_i - \frac{1-F_i(x)}{f_i(x)} = x_i - \frac{1}{\lambda_i}$. Given the bids X_1, X_2 , the seller examines $X_1 - \frac{1}{\lambda_1}, X_2 - \frac{1}{\lambda_2}$ and 0. If $X_i - \frac{1}{\lambda_i}$ is the largest of these three values, then bidder i gets the object and pays $\frac{1}{\lambda_i} + \left(X_{-i} - \frac{1}{\lambda_{-i}}\right)_+$. Bidders not receiving the object pay zero. The expected payoff of the seller is given by

$$\begin{aligned} R_{opt} &= \mathbb{E} \left[\max \left\{ X_1 - \frac{1}{\lambda_1}, X_2 - \frac{1}{\lambda_2}, 0 \right\} \right] \\ &= \int_0^\infty \mathbb{P} \left\{ \max \left\{ X_1 - \frac{1}{\lambda_1}, X_2 - \frac{1}{\lambda_2} \right\} \geq t \right\} dt \\ &= \int_0^\infty e^{-\lambda_1(t+1/\lambda_1)} + e^{-\lambda_2(t+1/\lambda_2)} - e^{-\lambda_1(t+1/\lambda_1)} e^{-\lambda_2(t+1/\lambda_2)} dt \\ &= \frac{e^{-1}}{\lambda_1} + \frac{e^{-1}}{\lambda_2} - \frac{e^{-2}}{\lambda_1 + \lambda_2}. \end{aligned}$$

- (b) Vickrey second price auction with a reserve price r . You don't need to find a closed form expression for the choice of r that maximizes the expected reward. However, for part (d) the optimal threshold should be used but it can be found numerically. As mentioned above, if the item is not sold, the value to the seller is zero, even if $r \neq 0$.

Solution: We shall assume truthful bidding and, for brevity, use X_i , without the hat, to denote the bid of bidder i . Given the bids X_1, X_2 , the seller examines X_1, X_2 and r . If $X_i \geq \max\{r, X_{-i}\}$ then bidder i gets the object for payment $\max\{r, X_{-i}\}$. If $\max\{X_1, X_2\} < r$ then buyer does not sell the object. Bidders not receiving the object pay zero.

Here is one of several ways to calculate R_v . The seller gets payoff at least r on the event $\{X_1 \geq r\} \cup \{X_2 \geq r\}$. The amount of payoff the seller gets above r is $(\min\{X_1, X_2\} - r)_+$.

Using the fact that $\min\{X_1, X_2\}$ has the exponential distribution with parameter $\lambda_1 + \lambda_2$, we find

$$\begin{aligned} R_v &= r\mathbb{P}\{X_1 \geq r \text{ or } X_2 \geq r\} + \mathbb{E}[(\min\{X_1, X_2\} - r)_+] \\ &= r \left[e^{-\lambda_1 r} + e^{-\lambda_2 r} - e^{-(\lambda_1 + \lambda_2)r} \right] + \frac{e^{-(\lambda_1 + \lambda_2)r}}{\lambda_1 + \lambda_2}, \end{aligned}$$

which we later numerically maximize with respect to r .

- (c) A sequential selling mechanism. The seller first takes a bid from bidder 1 and, depending on the bid, either sells the object to bidder 1 for some price, or does not sell to bidder 1. In the later case, the seller takes a bid from bidder 2 and, based on the bid, either sells for a price, or does not sell. Your mechanism should maximize the seller's expected payoff over all such sequential selling mechanisms. (Hint: Work backwards.)

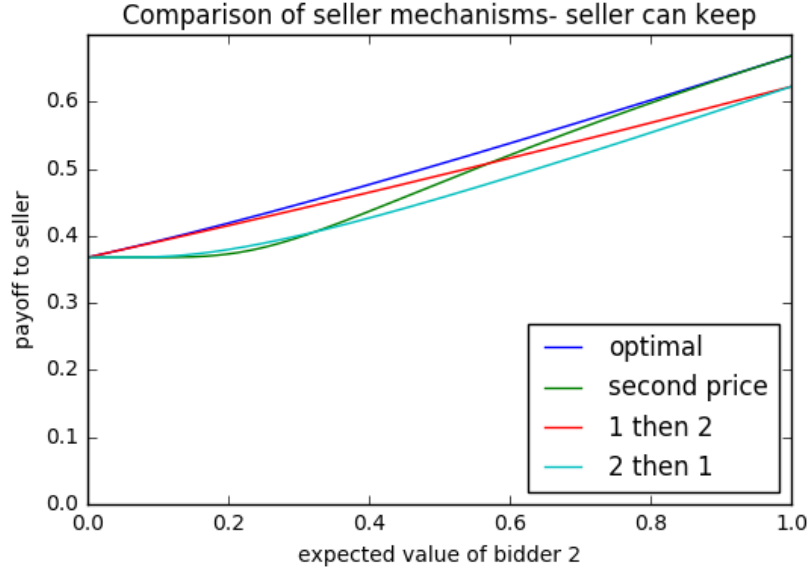
Solution: Following the hint, first consider what the seller does if the item is not sold to bidder 1. The bidder makes a decision by comparing $X_2 - \frac{1}{\lambda_2}, 0$. Thus, the bidder sells to buyer 2 if $X_2 \geq \frac{1}{\lambda_2}$ for the price of $\frac{1}{\lambda_2}$. Hence, given that the seller does not sell the item to bidder 1, the conditional expected payoff of the seller is $\mathbb{E} \left[\max \left\{ X_2 - \frac{1}{\lambda_2}, 0 \right\} \right] = \frac{e^{-1}}{\lambda_2}$.

This is the value of the item to the seller if the seller does not sell it to buyer 1 first. So when buyer 1 submits the bid X_1 , the seller compares $X_1 - \frac{1}{\lambda_1}$ and $\frac{e^{-1}}{\lambda_2}$. Thus, if $X_1 \geq \frac{1}{\lambda_1} + \frac{e^{-1}}{\lambda_2}$, the seller sells to bidder 1 for payment $\frac{1}{\lambda_1} + \frac{e^{-1}}{\lambda_2}$. Otherwise the seller does not sell to bidder 1. The expected payoff of the seller for the entire sequential mechanism is given by (arguing as in part (a)):

$$\begin{aligned} R_{12} &= \mathbb{E} \left[\max \left\{ X_1 - \frac{1}{\lambda_1}, \frac{e^{-1}}{\lambda_2} \right\} \right] \\ &= \frac{e^{-1}}{\lambda_2} + e^{-\lambda_1 \left(\frac{e^{-1}}{\lambda_2} + \frac{1}{\lambda_1} \right)} \frac{1}{\lambda_1} \\ &= \frac{e^{-1}}{\lambda_2} + \frac{e^{-1 - e^{-1} \lambda_1 / \lambda_2}}{\lambda_1}. \end{aligned}$$

- (d) (This part will be worth only 2 points, so skip if it you are not interested in programming.) Let R_{opt} , R_v , and R_{12} denote the expected seller's payoff for the mechanisms of (a)-(c) respectively, and let R_{21} denote the expected payoff for part (c) if, instead, bidder 2 participates first. Plot all four expected payoffs in one plot vs. $E[X_2]$ (equal to $1/\lambda_2$) for $0 < E[X_2] \leq 1$ and $E[X_1] = 1$.

Solution:



The plots illustrate the fact that the second price auction for the symmetric case $\lambda_1 = \lambda_2$ is optimal. The performance of all four mechanisms is pretty similar.

2. [A comparison of seller mechanisms when seller must sell.]

Repeat problem 1 under the assumption that the seller must sell the object. In other words, the value of the object to the seller is $-\infty$. Let R_v^o , R_{opt}^o , R_{12}^o , and R_{21}^o denote the respective expected payoffs. For the payoff optimal mechanism, assume that $\lambda_1 \leq \lambda_2$. (Hint: In part (a) you should find, for $\lambda_1 \leq \lambda_2$, that $R_{opt}^o = e^{-1+\lambda_1/\lambda_2} \left[\frac{1}{\lambda_1} - \frac{1}{\lambda_1+\lambda_2} \right]$.)

Solution: (a) Optimal mechanism Given the bids X_1, X_2 , the seller examines $X_1 - \frac{1}{\lambda_1}, X_2 - \frac{1}{\lambda_2}$. If $X_i - \frac{1}{\lambda_i}$ is the larger of these two values, then bidder i gets the object and pays $\left(\frac{1}{\lambda_i} + X_{-i} - \frac{1}{\lambda_{-i}} \right)_+$. Let us henceforth assume $\lambda_1 < \lambda_2$. The expected payoff of the seller is given by

$$R_{opt}^o = \mathbb{E} \left[\max \left\{ X_1 - \frac{1}{\lambda_1}, X_2 - \frac{1}{\lambda_2} \right\} \right]$$

One way to do the integration is to use the area rule for expectations. The first integral (area above the CDF on the positive axis) is R_{opt} , found in the previous problem, The second integral is

$$\begin{aligned} & \int_{-\frac{1}{\lambda_2}}^0 \mathbb{P} \left\{ \max \left\{ X_1 - \frac{1}{\lambda_1}, X_2 - \frac{1}{\lambda_2} \right\} \leq t \right\} dt \\ &= \int_{-\frac{1}{\lambda_2}}^0 \left(1 - e^{-1-\lambda_1 t} \right) \left(1 - e^{-1-\lambda_2 t} \right) dt \\ &= \int_{-\frac{1}{\lambda_2}}^0 1 - e^{-1-\lambda_1 t} - e^{-1-\lambda_2 t} + e^{-2-(\lambda_1+\lambda_2)t} dt, \end{aligned}$$

which can be further developed by doing the integrations. Some terms cancel with terms in

R_{opt} , yielding the final expression for R_{opt}^o given in the hint:

$$R_{opt}^o = e^{-1+\lambda_1/\lambda_2} \left[\frac{1}{\lambda_1} - \frac{1}{\lambda_1 + \lambda_2} \right].$$

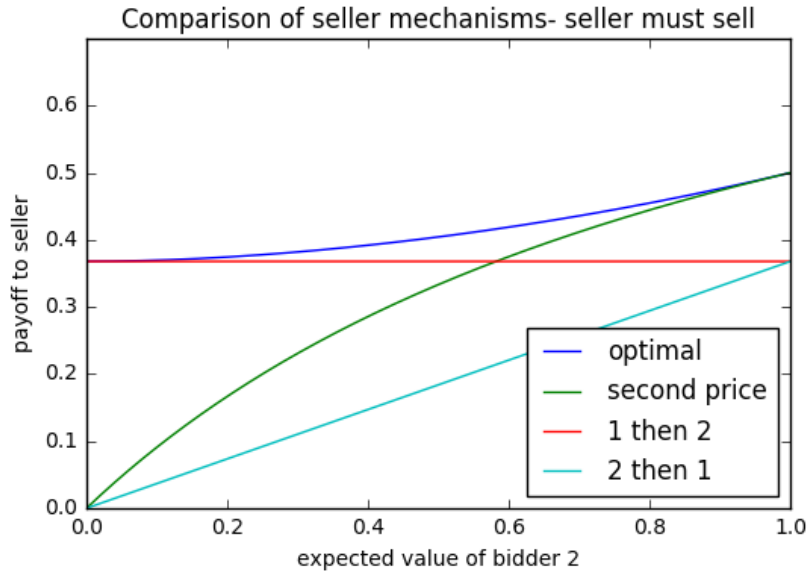
(b) Vickrey auction. Since the seller must sell, the reserve price is set to zero. So this is the same as part (a) of previous problem with $r = 0$. Seller sells to highest bidder for price $\min\{X_1, X_2\}$, resulting in expected payoff $R_v^o = \mathbb{E}[\min\{X_1, X_2\}] = \frac{1}{\lambda_1 + \lambda_2}$.

(c) Sequential selling mechanism. First consider what the seller does if the item is not sold to bidder 1. Since the seller can't afford to keep the item, the seller must sell the item to bidder 2 for price 0. (Although it is overkill, we could derive this from Myerson's theory. Given X_2 we consider the maximum of $X_2 - \frac{1}{\lambda_2}$ and $-\infty$, which is always achieved by the first term. The minimum allowable value X_2 could bid and still win is thus 0, which is the minimum to win payment.) The equivalent effect of bidder 2 is that it gives the seller free disposal of the item when meeting bidder 1.

So when buyer 1 submits the bid X_1 , the seller compares $X_1 - \frac{1}{\lambda_1}$ and 0. Thus, if $X_1 \geq \frac{1}{\lambda_1}$ then the seller sells to bidder 1 at price $\frac{1}{\lambda_1}$ and otherwise the seller does not sell the item to bidder 1. The expected payoff of the seller for the entire sequential mechanism is given by:

$$R_{12}^o = \mathbb{E} \left[\max \left\{ X_1 - \frac{1}{\lambda_1}, 0 \right\} \right] = \frac{e^{-1}}{\lambda_1}$$

(d) Plot



The plots again illustrate the optimality of second price auction in the symmetric case, $\lambda_1 = \lambda_2$. Both the second price auction and the 2 then 1 sequential mechanism have expected payoff converging to zero as $\mathbb{E}[X_2] \rightarrow 0$. Comparing to the plots in the previous problem shows a significant drop due to needing to sell the object. The difference will typically be smaller for a larger number of bidders.

3. [The core of a flow game]

We first review the max flow min cut theorem. A flow network (V, E, c, s, t) is a directed graph with vertex set V , edge set E , nonnegative edge capacities $c = (c_e : e \in E)$, and distinct vertices $s, t \in V$ representing the source and terminus of flow. Given a vector $(f_e)_{e \in E}$, the net flow into a vertex v is $\sum_{e \in E_{in}(v)} f_e - \sum_{e \in E_{out}(v)} f_e$. An $s - t$ flow is a vector $(f_e)_{e \in E}$ satisfying the capacity constraints $0 \leq f_e \leq c_e$ for all $e \in E$ such that the net flow into v is zero for any $v \in V \setminus \{s, t\}$. The value of an $s - t$ flow is the net flow into vertex t . An $s - t$ cut is a partition $C = (V_1, V_2)$ of V with $s \in V_1$ and $t \in V_2$ and the capacity of such a cut is $\sum_{e \in X_C} c_e$, where X_C , the cutset of C , is the set of edges in E going from a vertex in V_1 to a vertex in V_2 . The value of any flow is less than or equal to the capacity of any cut (weak duality) and the celebrated max-flow min-cut theorem is that strong duality holds. It is an instance of strong duality for linear programming (e.g. Problem 1, problem set 2, Fall 2017.)

Given a flow network (V, E, c, s, t) and a partition of E into disjoint sets $(E_i)_{i \in I}$ (written $E = \sqcup_{i \in I} E_i$), the flow game (I, v) is the cooperative game with set of players I and value function v such that for $S \subset I$, $v(S)$ is the maximum flow value for the network $(N, \sqcup_{i \in S} E_i, c, s, t)$. Show that the core of a flow game (I, v) is nonempty. (Hint: Think about how a payoff vector can be associated with a minimum value cut for the network with link set E .)

Solution: Let C^* be a minimum value cut for (V, E, c, s, t) , with associated cut set X_{C^*} . We claim that $(x_i : i \in I)$ is in the core of the flow game, where $x_i = \sum_{e \in E_i \cap X_{C^*}} c_e$. The inequality $\sum_{i \in S} x_i \geq v(S)$ holds for any $S \subset I$ by weak duality for the network with link set $\sqcup_{i \in S} E_i$. Equality when S is the grand coalition, $S = I$, holds by the max flow min cut theorem for the original network.

4. [Shoe market]

Consider the following market with transferable utilities, where $1 \leq n_1 \leq n_2$. Agents are indexed by $I = \{0\} \cup I_1 \cup I_2$ where $I_1 = \{1, \dots, n_1\}$ and $I_2 = \{n_1 + 1, \dots, n_1 + n_2\}$. There are $\ell = 2$ divisible goods, namely, units of left shoe and units of right shoe (Ignore the discrete nature of shoes!) The endowments are $w_0 = (0, 0)$, $w_i = (1, 0)$ for $i \in I_1$, and $w_i = (0, 1)$ for $i \in I_2$. The utility functions are given by $f_0(z_0) = \min\{z_{0,0}, z_{0,1}\}$ and $f_i(z_i) \equiv 0$ for $i \neq 0$. In words, agent 0 has unit value per pair of shoes, and the other agents each have a unit of shoe. If $n_1 < n_2$, there is less left shoe than right shoe.

- (a) Find the core, the payoff vector(s) for all competitive equilibria, and the Shapley value for the special case $n_1 = 1$ and $n_2 = 2$.

Solution: $v(S) = 1$ if S is the set of all agents with possibly either agent 2 or 3 omitted, and $v(S) = 0$ otherwise. Thus, the core constraints are the following (omitting redundant ones for some sets S with $v(S) = 0$):

$$\begin{aligned} x_i &\geq 0 \text{ for } 0 \leq i \leq 3 \\ x_0 + x_1 + x_2 &\geq 1 \\ x_0 + x_1 + x_3 &\geq 1 \\ x_0 + x_1 + x_2 + x_3 &= 1 \end{aligned}$$

which imply that $x_2 = x_3 = 0$. Thus, the core is $\{(a, 1 - a, 0, 0) : 0 \leq a \leq 1\}$.

The competitive equilibrium $(z^*, p^*) = ((z_{i,j}^*)_{i \in \{0,1,2,3\}, j \in \{1,2\}}, (p_1^*, p_2^*))$ satisfies $z^* \geq 0$. $\sum_{i=0}^3 z_i^* = (1, 2)$ and $z_i^* \in \arg \max_{z_i \geq 0} f_i(z_i) - p^*(z_i - w_i)$. In order that z_1^*, z_2^*, z_3^* be finite it is necessary that $p^* \geq 0$. In order that z_0^* be finite it is necessary that $p_1^* + p_2^* \geq 1$. If

$p_1^* + p_2^* > 1$ then $z_0^* = (0, 0)$ which implies that $z_{i,0}^* > 0$ for some $i \in \{1, 2, 3\}$ so $p_1^* = 0$ and similarly $p_2^* = 0$, yielding a contradiction, so $p_1^* + p_2^* = 1$. Some player must have an excess of right shoe so that $p_2^* = 0$. Thus for any competitive equilibrium, $f_0(z_0^*) = 1$ and $p^* = (1, 0)$. (The price vector is uniquely determined, the extra unit of right shoe can be distributed among the players arbitrarily.) The payoff vector at any competitive equilibrium is thus $(0, 1, 0, 0)$. In other words, all the value goes to player 1.

If players arrive in uniformly distributed random order, player 3 generates reward 1 if it arrives third and player 4 arrives last, which has probability $1/12$. Similarly for player 4. Players 0 and 1 have equal status for this choice of n_1 and n_2 so they get equal rewards for the Shapley value. Hence, the Shapley value is $(5/12, 5/12, 1/12, 1/12)$.

- (b) Find the core and competitive equilibrium (Shapley value not needed) in case $1 \leq n_1 < n_2$. Provide a brief explanation—full details not needed. Do your answers seem fair to you if $n_1 = 999$ and $n_2 = 1000$.

Solution: Similarly to part (a), we find $x_i = 0$ for $i \in I_2$. Also, if an agent in I_2 is dropped from the grand coalition, the value can drop by at most one, so $0 \leq x_i \leq 1$ for $i \in I_2$. Thus, the core is the set of vectors of the form $(n_1 - x_1 - \dots - x_{n_1}, x_1, \dots, x_{n_1}, 0, \dots, 0)$ such that $x_i \in [0, 1]$ for $1 \leq i \leq n_1$.

Similarly to part (a), (z^*, p^*) is a competitive equilibrium if and only if $p^* = (1, 0)$ and Z^* is feasible such that $f_0(z_0^*) = n_1$. In other words, player 0 is allocated all the left shoe and enough right shoe to make n_1 units of shoe, and the remaining $n_2 - n_1$ units of right shoe can be distributed among the players arbitrarily. The payoff vector for any competitive equilibrium is $(0, 1, \dots, 1, 0, 0, \dots, 0)$. This does not seem fair in case $n_1 = 999$ and $n_2 = 1000$. Just because there is one more right shoe player than left shoe player, all the right shoe players get zero payoff.

- (c) Find the core and competitive equilibrium in case $n_1 = n_2 = n$ for some $n \geq 1$. Provide a brief explanation—full details not needed.

Solution: Let x be in the core. Then $x_0 + \dots + x_{2n} = n$. Also, $v(S) \geq 0$ for all S so $x_i \geq 0$ for $0 \leq i \leq 2n$. If any agent in I_1 and any agent in I_2 is removed from the grand coalition, the value of the grand coalition drops by at most one. Therefore, $\max_{i \in I_1} x_i + \max_{j \in I_2} x_j \leq 1$. The core is the set of all x satisfying those constraints.

The competitive equilibria (p^*, z^*) is given by $p^* = (a, 1 - a)$ for some $a \in [0, 1]$ and z^* is the unique feasible vector with $f_0(z^*) = n$, namely, $z^* = ((n, n), (0, 0), \dots, (0, 0))$. Thus, the set of payoff vectors of the competitive equilibria is the set of vectors of the form $(0, a, \dots, a, 1 - a, \dots, 1 - a)$ for $0 \leq a \leq 1$. Interestingly, the payoff of the seller is zero. The seller pays unit value for each pair of shoes, and each pair of shoes has unit value to the seller. If the price for shoes were less than one, the seller would want to buy infinitely many pairs of shoes.