

ECE 586GT: Problem Set 2: Problems and Solutions

Uniqueness of Nash equilibria, zero sum games, evolutionary dynamics

Due: Tuesday, Sept. 25, at beginning of class**Reading:** Course notes, Sections 1.5-1.7 and Chapter 2

1. [Existence and uniqueness of NE for games with continuous type strategies]

This problem concerns an n player game with strategy space $S_i = (0, +\infty)$ for all players i , and space of strategy profiles $S = S_1 \times \cdots \times S_n$.

- (a) Consider the payoff functions $u_i(s) = (\ln s_i) - s_i s_{\text{tot}}$, where $s_{\text{tot}} = s_1 + \cdots + s_n$. Does there exist a pure strategy Nash equilibrium? If so, is it unique? Can you find an explicit expression for it?

Solution: For $i \in [n]$, for any s_{-i} fixed, u_i is a strictly concave function of s_i with limit $-\infty$ at 0 and $+\infty$. Hence, for each s_{-i} fixed there is a unique best response which is obtained by setting $\frac{\partial u_i}{\partial s_i} = 0$. Note that $s_{\text{tot}} = |s_{-i}| + s_i$, where $|s_{-i}| = \sum_{j \in [n], j \neq i} s_j$. Writing $u_i(s) = \ln s_i - s_i^2 - s_i |s_{-i}|$ we find $\frac{\partial u_i}{\partial s_i} = 0$ is equivalent to

$$\frac{1}{s_i} - 2s_i - |s_{-i}| = 0. \quad (1)$$

(Although it isn't needed, we note (1) implies that the best response function for any player i is $B_i(s_{-i}) = \frac{-|s_{-i}| + \sqrt{|s_{-i}|^2 + 8}}{4}$.)

Using the fact $s_{\text{tot}} = |s_{-i}| + s_i$, (1) can be rewritten as

$$\frac{1}{s_i} - s_i - s_{\text{tot}} = 0, \quad (2)$$

which can be used to determine s_i as a function of s_{tot} :

$$s_i = \frac{-s_{\text{tot}} + \sqrt{s_{\text{tot}}^2 + 4}}{2}$$

We can conclude that any Nash equilibrium (in pure strategies) is symmetric. Therefore, for a Nash equilibrium, $s_{\text{tot}} = ns_i$ and so

$$s_i = \frac{-ns_i + \sqrt{n^2 s_i^2 + 4}}{2}$$

Thus, there is a unique Nash equilibrium given by $s_i = \frac{1}{\sqrt{n+1}}$ for all $i \in [n]$.

- (b) Consider the payoff functions $u_i(s) = (\ln s_i) - s_i s_{\text{tot}} + \epsilon \sum_j \cos(s_j)$ for some $\epsilon > 0$. Find a positive constant $\bar{\epsilon}$ such that there exists a Nash equilibrium if $0 \leq \epsilon < \bar{\epsilon}$. To avoid some tedious steps, you may assume that the strategy space for each player is restricted to the closed interval $[\delta, 1/\delta]$, such that δ can be chosen so small that any best response for any player is in the interior of the interval.

Solution: Note that $\frac{\partial^2 u_i(s)}{(\partial s_i)^2} = -\frac{1}{s_i^2} - 2 - \epsilon \cos(s_i) < 0$ if $\epsilon \leq 2$. so $u_i(s)$ is a concave function of s_i for s_{-i} fixed, if $\epsilon \leq 2$. Also, the payoff functions are continuous and the strategy spaces $[\delta, 1/\delta]$ are compact, so there exists a pure strategy NE if $\epsilon \leq 2$ by the Debreu, Glicksberg, and Fan existence theorem.

- (c) For the payoff functions in part (b), find a positive constant $\bar{\epsilon}$ such that there exists a *unique* pure strategy Nash equilibrium if $0 \leq \epsilon < \bar{\epsilon}$. (Hint: A diagonal matrix with negative entries on the diagonal is negative definite, and the sum of a negative definite matrix and a negative semi-definite matrix is negative definite.)

Solution: We use the sufficient conditions for uniqueness in the notes based on the following matrix:

$$U(s) = \begin{pmatrix} \frac{\partial^2 u_1(s)}{(\partial s_1)^2} & \frac{\partial^2 u_1(s)}{\partial s_1 \partial s_2} & \cdots \\ \frac{\partial^2 u_2(s)}{\partial s_2 \partial s_1} & \ddots & \\ \vdots & & \end{pmatrix}$$

By direct calculation we find $U(s) = D(s) - J$, where $D(s)$ is the diagonal matrix $\text{diag}\left(-\frac{1}{s_i^2} - 1 - \epsilon \cos(s_i)\right)$ and J is the all ones $n \times n$ matrix. In particular, U is a symmetric matrix so we don't need to consider the symmetrization $U(s) + U^T(s)$. If $0 \leq \epsilon \leq 1$ then $D(s)$ is negative definite. Also, $-J$ is negative semidefinite. Thus, if $0 \leq \epsilon \leq 1$, $U(s)$ is negative definite, and there exists a unique Nash equilibrium,

2. [Market power in territory control game]

Recall the territory control game for a graph $G = (V, E)$ and $n \geq 2$ players, defined in problem set 1. The game is symmetric in the sense that the payoff functions are invariant under permutation of the players. A strategy profile (in either pure or mixed strategies) is symmetric if every player uses the same strategy.

- (a) Prove that for any symmetric finite player game, there exists a symmetric Nash equilibrium in mixed strategies. Thus, for the territory control game based on a connected graph $G = (V, E)$, there is a Nash equilibrium such that the expected payoff of each player is $\frac{|V|}{n}$.

Solution: This follows from the same proof as Nash's theorem.

- (b) Show that a two-player game such that the sum of payoffs is constant for all strategy pairs is equivalent to a zero sum two-player game. In particular, there exists a saddle point and a value of the game for each player.

Solution: Let u_{tot} denote the sum of payoffs, which is assumed to be independent of the strategy profile. We could impose a tax in the amount u_{tot} to player 1, or impose a tax of u_{tot}/n to every player, to make the sum of payoffs always equal to zero.

- (c) Suppose there are three players, but players 2 and 3 team up and split their payoffs with each other. Thus, players 2 and 3 together can be viewed as a super player; from the perspective of player 1, the game becomes a two-player game, with constant sum of payoffs $|V|$. Player 1 could be placing a Burger King and players 2 and 3 together place two McDonalds. Show that the value of the modified game for player 1 is greater than or equal to $|V|/4$ for any connected graph G . (Hint: What if there were two super players, each selecting nodes for two restaurants?)

Solution: If there were two super players, each selecting two nodes using a mixed strategy, then by symmetry the value of the game would be $|V|/2$ for each of the players. Transform the strategy of one of those super players by having the player first select its two nodes (possibly the same one twice), and then flipping a fair coin to determine which of those choices to keep, and which to discard. The mean payoff generated by the randomly retained node was at least $|V|/4$ before the other randomly selected node

was dropped. The dropping of the other choice by the same player cannot decrease the expected points is generated by the retained node, which is thus at least $|V|/4$. For example, if the player had two Burger King restaurants, then closing one of them could generate more traffic for the other.

- (d) Consider the scenario of part (c) for the special case of a line graph $V = \{1, 2, \dots, m\}$, such that $m = 4b + 2$ for some nonnegative integer b , and such that $E = \{[i, i + 1] : 1 \leq i \leq m - 1\}$. Show that the value of the modified game for player 1 is equal to $|V|/4$. (Hint: By part(c) only an upper bound on the average payoff of player 1 is needed.)

Solution: Suppose player 2 selects $b + 1$ and player 3 selects $3b + 2$. Then the payoff of player 1 is less than or equal to $m/4$ for all possibilities. Specifically, it ranges from $(k + 1)/2$ to $|V|/4$ as s_1 increases from 0 to $b + 1$. It remains flat at $|V|/4$ for $b + 1 \leq s_1 \leq 3b + 2$.

3. [Dual of a transport problem]

Let $n \geq 1$ and let W be an $n \times n$ matrix with nonnegative elements. Consider the following linear optimization problem:

$$\begin{aligned} \min_X \quad & \sum_{i,j \in [n]} X_{i,j} W_{i,j} \\ \text{subject to:} \quad & \sum_{j \in [n]} X_{i,j} = 1 && \text{for } i \in [n] \\ & \sum_{i \in [n]} X_{i,j} = 1 && \text{for } j \in [n] \\ & X_{i,j} \geq 0 && \text{for } i, j \in [n] \end{aligned}$$

An interpretation is that there are n sources of some good, n destinations, $W_{i,j}$ is the cost of transporting a unit of good from i to j , and $X_{i,j}$ is the amount of good transported from i to j . Each source provides one unit of good and each destination receives one unit of good. The problem in vector matrix form can be written as:

$$\begin{aligned} \min_X \quad & \langle X, W \rangle \\ \text{subject to:} \quad & X\mathbf{1} = \mathbf{1}, \quad X^T\mathbf{1} = \mathbf{1}, \quad X \geq 0. \end{aligned}$$

- (a) Derive the dual problem by first finding the Lagrangian function. Find a simple version of the dual problem. (Hint: You can either eliminate the Lagrange multipliers for the constraints $X_{i,j} \geq 0$, or don't handle those constraints with Lagrange multipliers in the first place.

Solution: Using strong duality for linear programs, we find

$$\begin{aligned}
& \min_{X: X\mathbf{1}=\mathbf{1}, X^T\mathbf{1}=\mathbf{1}, X \geq 0} \langle X, W \rangle \\
&= \min_{X: X \geq 0} \max_{\mu^{(1)}, \mu^{(2)}} \langle X, W \rangle + (\mathbf{1} - X\mathbf{1})^T \mu^{(1)} + (\mathbf{1} - X^T\mathbf{1})^T \mu^{(2)} \\
&= \max_{\mu^{(1)}, \mu^{(2)}} \min_{X: X \geq 0} \langle X, W \rangle + (\mathbf{1} - X\mathbf{1})^T \mu^{(1)} + (\mathbf{1} - X^T\mathbf{1})^T \mu^{(2)} \\
&= \max_{\mu^{(1)}, \mu^{(2)}} \min_{X: X \geq 0} \mathbf{1}^T(\mu^{(1)} + \mu^{(2)}) + \langle X, W - \mu^{(1)}\mathbf{1}^T - \mathbf{1}(\mu^{(2)})^T \rangle \\
&= \max_{\mu^{(1)}, \mu^{(2)}: \mu^{(1)}\mathbf{1}^T + \mathbf{1}(\mu^{(2)})^T \leq W} \mathbf{1}^T(\mu^{(1)} + \mu^{(2)})
\end{aligned}$$

In other words the dual problem is

$$\begin{aligned}
& \max_{\mu^{(1)}, \mu^{(2)}} \sum_{i \in [n]} \mu_i^{(1)} + \sum_{j \in [n]} \mu_j^{(2)} \\
& \text{subject to:} \quad \mu_i^{(1)} + \mu_j^{(2)} \leq W_{i,j} \quad \text{for } i, j \in [n]
\end{aligned}$$

(An interpretation of the dual variables is that $\mu_i^{(1)}$ is a transport charge per unit good at source i and $\mu_j^{(2)}$ is a transport charge per unit good at destination j . The dual problem entails maximizing the sum of payments subject to a constraint showing the charges are justified by the transport costs. The equality constraints in the primal problem could be replaced by inequality constraints: $X\mathbf{1} \geq \mathbf{1}$, $X^T\mathbf{1} \geq \mathbf{1}$, and that would lead to nonnegativity constraints on the multipliers in the dual problem.

- (b) Find the common value of the primal and dual, and solutions, for $W = \begin{pmatrix} 3 & 8 \\ 2 & 4 \end{pmatrix}$.

Solution: $X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\mu^{(1)} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, $\mu^{(2)} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$, with value 7 each. (The solution to the dual problem is not unique. Adding any constant multiple of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ to $\mu^{(1)}$ and subtracting it from $\mu^{(2)}$ yields another solution.)

4. [Evolutionarily stable strategies and states]

Consider the following symmetric, two-player game::

	1	2
1	1,1	1,2
2	2,1	0,0

- (a) Does either player have a (weakly or strongly) dominant strategy?

Solution: No.

- (b) Identify all the pure strategy and mixed strategy Nash equilibria.

Solution: (1,2) and (2,1) are pure strategy NE. As usual, there is no NE in which only one strategy is pure. If (p, q) is an NE such that both p and q are nondegenerate mixed strategies, either action of player one must be a best response to q , so $1 = 2q_1$, or $q = (\frac{1}{2}, \frac{1}{2})$. Similarly, $p = (\frac{1}{2}, \frac{1}{2})$. Thus, $((\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}))$ is the unique NE in nondegenerate mixed strategies.

- (c) Identify all evolutionarily stable pure strategies and all evolutionarily stable mixed strategies.

Solution: Recall that if p is an ESS, then (p, p) is an NE. The pure strategy NEs found in part (b) are not symmetric, so there are no pure ESSs. It remains to see whether the mixed strategy NE p given by $p = (\frac{1}{2}, \frac{1}{2})$ is an ESS. By definition, we need to check whether for any $p' \neq p$, either (i) $u(p', p) < u(p, p)$, or (ii) $(u(p', p) = u(p, p) \text{ and } u(p, p') > u(p', p'))$. Since $u(p', p) = u(p, p)$ for all choices of p' , the question comes down to whether $u(p, p') > u(p', p')$ for all $p' \neq p$. That is, whether $\frac{1}{2} + \frac{1}{2}(2)p'_1 > p'_1 + (1 - p'_1)(2p'_1)$ for all $p' \neq p$. Or, equivalently, whether $2(p'_1 - \frac{1}{2})^2 > 0$ for $p \neq p'$. This condition is true, so $(\frac{1}{2}, \frac{1}{2})$ is a mixed ESS.

- (d) The replicator dynamics based on this game represents a large population consisting of type 1 and type 2 individuals. Show that the evolution of the population share vector $\theta(t)$ under the replicator dynamics for this model reduces to a one dimensional ordinary differential equation for $\theta_t(1)$, the fraction of the population that is type 1.

Solution: The replicator dynamics for the population share vector θ_t are given by $\dot{\theta}_t(a) = \theta_t(a)(u(a, \theta_t) - u(\theta_t, \theta_t))$ for $a \in \{1, 2\}$. Although there are two equations, this system is actually one dimensional because $\theta_t(2) = 1 - \theta_t(1)$. Let $x_t = \theta_t(1)$. Then $u(1, \theta_t) \equiv 1$, and $u(\theta_t, \theta_t) = x_t^2 + 3x_t(1 - x_t) = 3x_t - 2x_t^2$. So the replicator dynamics become $\dot{x}_t = x_t(1 - 3x_t + 2x_t^2)$, or, equivalently,

$$\dot{x}_t = x_t(1 - x_t)(1 - 2x_t). \quad (3)$$

- (e) Identify the steady states of the replicator dynamics.

Solution: The right hand side of (3) is zero for $x_t \in \{0, \frac{1}{2}, 1\}$, so there are three steady states for the replicator dynamics: $(1, 0)$, $(\frac{1}{2}, \frac{1}{2})$, and $(0, 1)$.

- (f) Of the steady states identified in the previous part, which are asymptotically stable states of the replicator dynamics? Justify your answer.

Solution: Of the three steady states, only $(\frac{1}{2}, \frac{1}{2})$ is asymptotically stable. If $x_0 = \epsilon$ for an arbitrarily small but positive ϵ , then $\dot{x}_t > 0$ and x will converge monotonically up to $\frac{1}{2}$, so 0 is not even a stable steady state (so it is not asymptotically stable). Similarly, 1 is not a stable steady state. However, $\frac{1}{2}$ is an asymptotically stable steady state of x because the right hand side of (3) has a down crossing of zero at $\frac{1}{2}$. Hence $(\frac{1}{2}, \frac{1}{2})$ is an asymptotically stable state for the replicator dynamics. (Another justification for this problem can be given by applying general facts about ESSs and replicator dynamics. States $(1, 0)$ and $(0, 1)$ can't be asymptotically stable, or even stable, states for the replicator dynamics, because, when played against themselves, they don't give NEs. The mixed state $(\frac{1}{2}, \frac{1}{2})$ is an asymptotically stable state of the replicator dynamics because it is an ESS.)