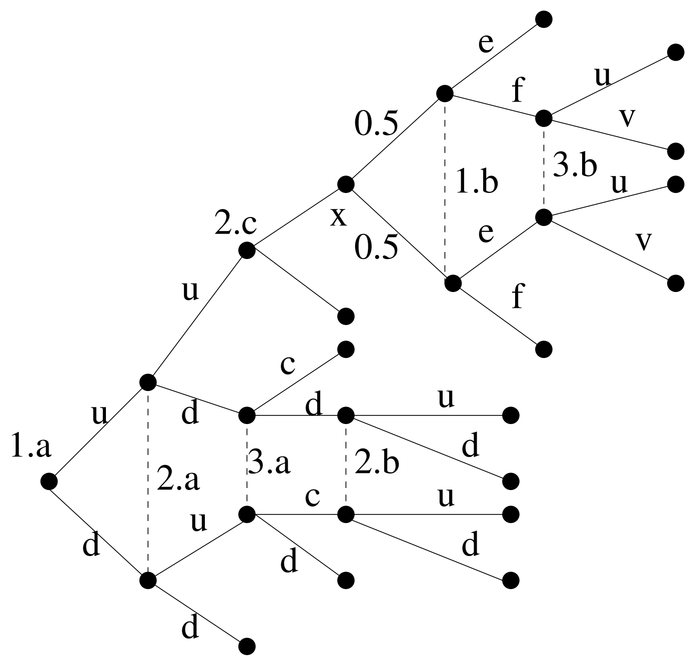
University of Illinois Fall 2017

ECE 586GT: Exam II

Tuesday, December 12, 2017
7:00 p.m. — 8:30 p.m.
2013 Electrical Engineering Building

1. [**15 points**] Consider the three player sequential game with imperfect information shown.



(Node x is controlled by nature, and the payoffs at the leaves are not shown.) Identify which player(s) 1, 2, 3 do/does *not* have perfect recall, and explain why.

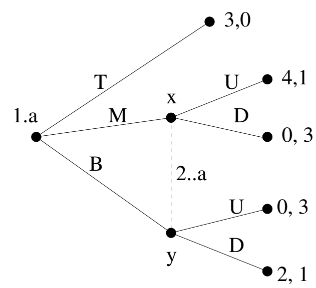
Solution: Player 2 does not have perfect recall, because at information set 2.b the player cannot perfectly recall what action he/she took at information state 2.a. (Players 1 and 3 have perfect recall.)

2. [**10 points**] Suppose Myerson's optimal selling mechanism is used by a seller to sell an object to a set of three bidders, using the min to win payment rule. Suppose the virtual valuation functions are given by $\psi_1(x_1) = x_1 - 1$, $\psi_2(x_2) = x_2 - 2$, and $\psi_3(x_3) = x_3 - 3$, and the value of the object to seller is zero (i.e. free disposal assumption holds). For each of the bid profiles given below, determine which bidder, if any, gets the object, and for what payment.

   (a) Bid profile $(6.1, 5.8, 7.9)$.
   Solution: The vector of virtual values is $(5.1, 3.8, 4.9)$, so bidder 1 gets the object. The payment is 5.9 because 5.9 is the minimum value of x_1 such that $\psi_1(x_1) \geq 4.9$.

   (b) Bid profile $(0.5, 1.9, 1.6)$.
   Solution: All the virtual values are negative, so no bidder gets the object and no bidder makes a payment.

3. [**25 points**] Consider the two player sequential game with imperfect information shown.



(a) (9 points) Give the normal form version of the game, and identify all the Nash equilibria in pure strategies (if any).

Solution:

	U	D
T	3,0	3,0
M	4,1	0,3
B	0,3	2,1

(T, D) is the unique pure strategy Nash equilibrium.

(b) (8 points) Identify all the Nash equilibria $\sigma = (\sigma_1, \sigma_2)$ in mixed strategies (Hint: There is more than one.)

Solution: A strategy profile σ is a Nash equilibrium if and only if it has the form $([T], q[U] + (1-q)[D])$ for some q with $0 \le q \le 3/4$. In other words. player 2 selects U with probability q and D with probability $1-q$.

To arrive at the above answer, reason as follows. Suppose (σ_1, σ_2) is a Nash equilibrium. Since T dominates B for player 1, σ_1 must assign zero probability to B. If σ_1 assigns positive probability to M then $\sigma_2 = [D]$, so the expected payoff of player 1 is 3 for action T and 0 for action M, giving a contradiction. Since all alternatives are eliminated, it must be that σ_1 is the pure strategy $[T]$. For $\sigma_1 = [T]$, any response for player 2 is a best response. In order for $[T]$ to be a best response for player 1, player 2 can use any mixed strategy of the form $q[U] + (1-q)[D])$ with $0 \le q \le 3/4$.

(c) (8 points) Identify all the sequential equilibria of the game. (Hint: Recall that a sequential equilibrium is an assessment (σ, μ) that is sequentially rational and consistent.)

Solution: Suppose (σ, μ) is a sequential equilibrium. It follows that σ is a Nash equilibrium, so assume σ has the form of a Nash equilibrium with parameter q found in part (b). It remains to determine what choices of beliefs μ, if any, would make (σ, μ) a sequential equilibrium. Since information set $2.a$ is the only one with more than one state, specifying μ is equivalent to specifying the belief of player 2, $\mu(2.a)$.

If $q \neq 0$ then the belief of player 2 must be such that player 2 believes either action gives the same expected payoff, which is true if and only if $\mu(x|2.a) = \mu(y|2.a) = 0.5$. The other possibility is $\sigma = (T, D)$ (i.e. $q = 0$). Action D is a best response for player 2 if and only if player 2 believes that the expected payoff for action D is greater than or equal to the expected payoff for action T. That is true if and only if $(\mu(x|2.a), \mu(y|2.a)) = (p, 1-p)$ for some $p$with $0.5 \le p \le 1$.

In summary, the sequential equilibria are the assessments (σ, μ) of one of the following two forms:

$$\left(([T], q[U] + (1-q)[D]), \frac{1}{2}[x] + \frac{1}{2}[y]\right) \qquad 0 \le q \le \frac{3}{4}$$

$$((([T], [D]), p[x] + (1-p)[y]) \qquad 0.5 \le p \le 1$$

4. **[25 points]** Consider the normal form game $G = (I, (S_i)_{i \in I}, (u_i)_{i \in I})$ such that $I = \{1, 2, 3\}$, $S_i = \{H, T\}$, and $u_i = |\{j : s_j \neq s_i\}|$, for all $i \in I$. In other words, there are three players, each selecting H or T, and the payoff of player i is the number of other players selecting an action different from the action selected by i.

(a) (13 points) Find the minmax value vector $\underline{v}$ (defined by $\underline{v}_i = \min_{\sigma_{-i}} \max_{\sigma_i} u_i(\sigma)$ for $1 \leq i \leq 3$.)

Solution: If player i selects H with probability 1/2 and T with probability 1/2 (i.e. fair coin flip) then her/his expected payoff is 1 no matter what strategies the other two players use. So $\underline{v}_i \geq 1$. If the other players also use the same strategy, then the payoff of player i is 1, no matter what strategy i uses, so $\underline{v}_i \leq 1$. Thus, $\underline{v}_i = 1$, and so $\underline{v} = (1, 1, 1)$.

(b) (12 points) Find the Nash realization region for repeated play of G with discount factor δ sufficiently close to one. (It consists of the feasible, strictly individually rational payoff profiles of G.)

Solution: A payoff vector (v_1, v_2, v_3) is feasible if it is in the convex hull of the set of payoff vectors for pure strategies: $\{(2, 1, 1), (1, 2, 1), (1, 1, 2), (0, 0, 0)\}$. Equivalently, the set of feasible payoff profiles is given by $\{v \in \mathbb{R}^3 : v_1 + v_2 + v_3 \leq 4, v_2 + v_3 \leq 3v_1, v_1 + v_3 \leq 3v_2, v_1 + v_2 \leq 3v_3\}$. Intersecting with $\{v : v_i > \underline{v}_i\}$ yields that the Nash realization region is $\{v : v_1 + v_2 + v_3 \leq 4, v_i \geq 1 \text{ for } 1 \leq i \leq 3\}$.

(Note: Since $\underline{v}$ is also the payoff vector of a Nash equilibrium (namely, the equilibrium such that each player uses H or T with probability 1/2), the region is realizable in subgame perfect equilibrium for the repeated game by Friedman's theorem.)

5. **[25 points]** Consider the market with transferable utilities $M = (I, \ell, (w_i), (f_i))$ such that $I = \{0, 1, \ldots, m\}$ for some $m \geq 1$, $\ell = 1$, and

$$f_0(z_0) = z_0^{1/2} \qquad w_0 = 0$$
$$f_i(z_i) = 0 \qquad w_i = 1, \text{ for } i \neq 0$$

Player 0 represents a factory owner and the other players represent workers, each with one unit of work to contribute. (This is a variation of an example considered in the notes. Here work is considered to be a divisible good.)

(a) (12 points) Find the core of the induced cooperative game.

Solution: (10 points) The coalitional value function is given by

$$v(S) = \begin{cases} f(k) & \text{if } 0 \in S \text{ and } S \text{ contains } k \text{ workers} \\ 0 & \text{if } 0 \notin S \text{ or } S \text{ contains no workers} \end{cases}$$

This value function is the same when it is assumed that each worker in any coalition always contributes one unit of work, and in an example in the course notes. By the reasoning given there, the core is given by

$$\{(x_0, x_1, \ldots, x_m) : 0 \leq x_i \leq f(m) - f(m-1), x_0 + \cdots + x_m = f(m)\}.$$

(b) (13 points) Find the competitive equilibrium, (z^*, p^*), for the induced cooperative game, and identify the corresponding payoff profile x^*.

Solution: Only the factory owner values work, so $z^* = (m, 0, \ldots, 0)$. The response function of the factory to price $p > 0$ is $z_0(p) = \arg\max_{z \geq 0}(z^{1/2} - zp)$, which is achieved when $\frac{1}{2}z^{-1/2} = p$, giving $z_0(p) = \frac{1}{4p^2}$. In order for the factory to hire all of the workers, so that the supply of work is equal to the demand, requires $p^* = \frac{1}{2}m^{-1/2} = f'(m)$. Thus, $x^* = (f(m) - mp^*, p^*, \ldots, p^*) = (\frac{\sqrt{m}}{2}, \frac{1}{2\sqrt{m}}, \ldots, \frac{1}{2\sqrt{m}})$. Thus, under the competitive equilibrium, the factory owner and the group of all workers each receive half the value of the grand coalition.